Science, Technology and Innovation
ISSN 2544-9125 ▪ 2022; 16 (3-4): 50-56
doi: 10.55225/sti.478

STI • 2022;16 (3-4): 50-56

The composition operators between Morrey type spaces

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Original article

Abstract

In this paper, we characterize the boundedness of composition operator C_φ from Morrey space $H^2{}_{K_1}$ to $H^2{}_{K_2}$ on the unit complex disk.

Keywords

- composition operator
- Morrey type space
- boundedness

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1. Introduction

Let $\mathbb{D}$ be the open unit disk and $\mathbb{T}$ be the unit circle in the complex plane $\mathbb{C}$. We use ϕ to denote an analytic self-map of $\mathbb{D}$. The composition operator C_ϕ is defined by $C_\phi f = f \circ \phi$, $f \in H(\mathbb{D})$, where $H(\mathbb{D})$ is the space of all analytic functions in $\mathbb{D}$.

The Hardy space H^2 consists of analytic functions f in $\mathbb{D}$ satisfying

$$\sup_{0\le r<1} \frac{1}{2\pi}\int_0^{2\pi} |f(re^{i\theta})|^2 d\theta < \infty.$$

Given a function $K : [0,\infty) \to [0,\infty)$ be a right-continuous and nondecreasing function, we denote the Morrey space H^2_K of all analytic functions $f \in H^2$ on $\mathbb{D}$ such that

$$\|f\|_{H^2_K} = \left(\sup_{I\subseteq\mathbb{T}} \frac{1}{K(|I|)}\int_I |f(\zeta) - f_I|^2|d\zeta|\right)^{\frac{1}{2}} < \infty,$$

where the supremum is taken over all sub-arcs $I \subseteq \mathbb{T}$ with $|I|$ being their arc-lengths, and

$$f_I = \frac{1}{|I|}\int_I f(\zeta)|d\zeta|.$$

The space H^2_K was first introduced in [5]. It is easy to check that H^2_K is a Banach space under the norm $|f(0)| + \|f\|^2_{H_K}$. In the case of $K(t) = t^p$, $0 < p < 1$, the space H^2_K gives the analytic Campanato space CA_p (see[7]), especially, if $K(t) = t$, then the space H^2_K coincides with $BMOA$, the intersection of the Hardy space H^2 on $\mathbb{D}$ and $BMO(\mathbb{T})$, the space of functions of bounded mean oscillation on $\mathbb{T}$. The main results in this paper cover the corresponding $BMOA$ results in [2] and [4].

For $f \in H^2$, set

$$\|f\|_{H^2_K,*} = \sup_{a\in\mathbb{D}}\left\{\frac{1-|a|^2}{K(1-|a|^2)}\cdot\|f\circ\sigma_a - f(a)\|_2^2\right\}^{\frac{1}{2}}.$$

Article info

Article history

- Received: 2022-12-31
- Accepted: 2023-04-15
- Published: 2023-04-21

Publisher

University of Applied Sciences in Tarnow
ul. Mickiewicza 8, 33-100 Tarnow, Poland

Financing

This work is supported by Lingnan Normal University (Grant No. LY2202).

Conflict of interest

None declared.

An important tool in the study of H^2_K is the auxiliary function φ_K, defined by

$$\varphi_K(s) = \sup_{0<t\leq 1} \frac{K(st)}{K(t)} \quad 0 < s < \infty.$$

The condition

$$(1) \qquad \int_1^\infty \varphi_K(s)\frac{ds}{s^2} < \infty.$$

has played a crucial role in the study of the so-called Q_K type spaces. See [1], [6] for instance. This condition will be crucial for us to study H^2_K spaces as well.

We summarize the main results of the paper as below.

Theorem 1.1. Let ϕ be an analytic self-map on $\mathbb{D}$ and $K_i : [0,\infty) \to [0,\infty)$ $i = 1,2$ be right-continuous and nondecreasing functions satisfying condition (1). Then $C_\phi : H^2_{K_1} \to H^2_{K_2}$ is bounded if and only if

$$(2) \qquad \sup_{a\in\mathbb{D}} \left\{ \frac{1-|a|^2}{K_2(1-|a|^2)} \cdot \frac{K_1(1-|\phi(a)|^2)}{1-|\phi(a)|^2} \cdot \|\sigma_{\phi(a)} \circ \phi \circ \sigma_a\|_2^2 \right\} < \infty$$

where

$$\sigma_b(z) = \frac{b-z}{1-\bar{b}z}, \quad \|f\|_2 = \left(\int_{\mathbb{T}} |f(\xi)|^2 |d\xi|\right)^{\frac{1}{2}} \qquad b, z \in \mathbb{D}.$$

Theorem 1.2. Let ϕ be an analytic self-map on $\mathbb{D}$ and $K : [0,\infty) \to [0,\infty)$ be a right-continuous and nondecreasing function satisfying condition (1). Then $C_\phi : H^2_K \to H^2_K$ is always bounded with

$$\|C_\phi f\|_{H^2_K,*} \leq \left\{ \frac{1+|\phi(0)|}{1-|\phi(0)|} \cdot \varphi_K\left(\frac{1-|\phi(0)|}{1+|\phi(0)|}\right) \right\}^{\frac{1}{2}} \|f\|_{H^2_K,*}\ .$$

In what follows, we use the notation $A \lesssim B$ to indicate that there is a constant $C > 0$ with $A \leq CB$. A and B are called equivalent, denoted by "$A \simeq B$", if there exists some C such that $A \lesssim B \lesssim A$.

2. Proof of the main results

In this section, we will give the proofs of the main results. To this end, we first give some lemmas which would be used in the following.

Lemma 2.1. Suppose K is a right-continuous and nondecreasing function. Then $f \in H^2_K$ if and only if $\|f\|_{H^2_K,*} < \infty$.

Proof. According to Theorem 3.1 in [5], we know that $f \in H^2_K$ if and only if

$$\sup_{a\in\mathbb{D}} \frac{1-|a|^2}{K(1-|a|)} \int_{\mathbb{D}} |f'(z)|^2(1-|\sigma_a(z)|^2)dA(z) < \infty.$$

For $f \in H^2$, the well-known Littlewood-Paley identity gives

$$\|f - f(0)\|_2^2 = \quad 2\int_{\mathbb{D}} |f'(z)|^2 \ln\frac{1}{|z|^2} dA(z),$$

and

$$\|f - f(0)\|_2^2 \simeq \int_{\mathbb{D}} |f'(z)|^2(1-|z|^2)dA(z).$$

Replace f by $f \circ \sigma_a$, we have

$$\int_{\mathbb{D}} |f'(z)|^2 \ln\frac{1}{|\sigma_a(z)|^2} dA(z) \simeq \int_{\mathbb{D}} |f'(z)|^2(1-|\sigma_a(z)|^2)dA(z).$$

Meanwhile, for $a \in \mathbb{D}$, we know

$$K(1-|a|) \simeq K(1-|a|^2)$$

Therefore,

$$\begin{aligned}\|f\|^2_{H^2_K,*} &= \sup_{a\in\mathbb{D}}\left\{\frac{1-|a|^2}{K(1-|a|^2)}\cdot\|f\circ\sigma_a - f(a)\|_2^2\right\}\\ &= 2\sup_{a\in\mathbb{D}}\frac{1-|a|^2}{K(1-|a|^2)}\cdot\int_{\mathbb{D}}|f'(z)|^2\ln\frac{1}{|\sigma_a(z)|^2}dA(z)\\ &\simeq \sup_{a\in\mathbb{D}}\frac{1-|a|^2}{K(1-|a|)}\cdot\int_{\mathbb{D}}|f'(z)|^2(1-|\sigma_a(z)|^2)dA(z).\end{aligned}$$

This shows $\|f\|_{H^2_K,*} < \infty$. □

Lemma 2.2. Suppose K is a right-continuous and nondecreasing function satisfying condition (2). Let

$$f_b(z) = \left[\frac{K(1-|b|^2)}{1-|b|^2}\right]^{\frac{1}{2}}[\sigma_b(z) - b].$$

Then f_b is uniformly bounded in H^2_K , that is, for all $b\in\mathbb{D}$,

$$\|f_b\|_{H^2_K,*} < \infty.$$

Proof. A simple computation gives

$$\begin{aligned}|f_b'(z)|^2 &= \frac{K(1-|b|^2)}{1-|b|^2}\cdot|\sigma_b'(z)|^2\\ &= (1-|b|^2)K(1-|b|^2)|1-\bar{b}z|^{-4},\end{aligned}$$

and

$$1-|\sigma_a(z)|^2 = \frac{(1-|a|^2)(1-|z|^2)}{|1-\bar{a}z|^2}.$$

Using [8, Lemma 1], we get the following estimate:

$$\begin{aligned}&\|f_b\circ\sigma_a - f_b(a)\|_2^2\\ \simeq\ &\int_{\mathbb{D}}|f_b'(z)|^2(1-|\sigma_a(z)|^2)dA(z)\\ \simeq\ &(1-|a|^2)(1-|b|^2)K(1-|b|^2)\int_{\mathbb{D}}\frac{1-|z|^2}{|1-\bar{a}z|^2|1-\bar{b}z|^4}dA(z)\\ \lesssim\ &(1-|a|^2)(1-|b|^2)K(1-|b|^2)\frac{1}{(1-|b|^2)|1-\bar{a}b|^2}\\ =\ &\frac{(1-|a|^2)K(1-|b|^2)}{|1-\bar{a}b|^2}.\end{aligned}$$

Hence,

$$\begin{aligned}\|f_b\|^2_{H^2_K,*} &= \sup_{a\in\mathbb{D}}\frac{1-|a|^2}{K(1-|a|^2)}\|f_b\circ\sigma_a - f_b(a)\|_2^2\\ &\lesssim \sup_{a\in\mathbb{D}}\frac{(1-|a|^2)^2}{|1-\bar{a}b|^2}\cdot\frac{K(1-|b|^2)}{K(1-|a|^2)}\\ &\lesssim 1.\end{aligned}$$

Thus,we finish the proof of this lemma. □

Proof of Theorem 1.1: Assume that (2) holds. To show $C_\phi : H^2_{K_1} \to H^2_{K_2}$ is bounded, it suffices to verify $\|C_\phi f\|^2_{H^2_{K_2},*} \lesssim \|f\|^2_{H^2_{K_1},*}$.

Suppose $g(0) = 0 = \psi(0)$, here $g \in H^2$ and ψ is an analytic self-map on $\mathbb{D}$, it follows from [3, Proposition 2.3] that

$$\|g \circ \psi\|_2 \lesssim \|g\|_2 \|\psi\|_2 \ .$$

Taking

$$g_a = f \circ \sigma_{\phi(a)} - f \circ \phi(a), \ \psi_a = \sigma_{\phi(a)} \circ \phi \circ \sigma_a,$$

It is easy to check that $g_a \in H^2$, ψ_a is an analytic self-map on $\mathbb{D}$, and

$$g_a(0) = 0 = \psi_a(0).$$

thus we have

$$\|g_a \circ \psi_a\|_2 \lesssim \|g_a\|_2 \|\psi_a\|_2 \ . \tag{3}$$

Notice that

$$g_a \circ \psi_a = f \circ \phi \circ \sigma_a - f \circ \phi(a).$$

It follows from Lemma 2.1 and (3), we deduce

$$\begin{aligned}
&\|C_\phi f\|^2_{H^2_{K_2},*} \\
= \ & \sup_{a\in\mathbb{D}} \frac{1-|a|^2}{K_2(1-|a|^2)} \cdot \|f \circ \phi \circ \sigma_a - f \circ \phi(a)\|_2^2 \\
= \ & \sup_{a\in\mathbb{D}} \frac{1-|a|^2}{K_2(1-|a|^2)} \cdot \|g_a \circ \psi_a\|_2^2 \\
\lesssim \ & \sup_{a\in\mathbb{D}} \frac{1-|a|^2}{K_2(1-|a|^2)} \cdot \|g_a\|_2^2 \|\psi_a\|_2^2 \\
= \ & \sup_{a\in\mathbb{D}} \frac{1-|\phi(a)|^2}{K_1(1-|\phi(a)|^2)} \cdot \|f \circ \sigma_{\phi(a)} - f \circ \phi(a)\|_2^2 \\
& \cdot \frac{1-|a|^2}{K_2(1-|a|^2)} \cdot \frac{K_1(1-|\phi(a)|^2}{1-|\phi(a)|^2} \cdot \|\psi_a\|_2^2 \\
\lesssim \ & \|f\|^2_{H^2_{K_1},*} \cdot \sup_{a\in\mathbb{D}} \frac{1-|a|^2}{K_2(1-|a|^2)} \cdot \frac{K_1(1-|\phi(a)|^2)}{1-|\phi(a)|^2} \cdot \|\sigma_{\phi(a)} \circ \phi \circ \sigma_a\|_2^2 \ .
\end{aligned}$$

This shows C_ϕ is bounded from $H^2_{K_1}$ to $H^2_{K_2}$.

Conversely, for $f \in H^2$, we have

$$\int_{\mathbb{D}} |(C_\phi f)'(z)|^2 \ln \frac{1}{|z|} dA(z) = \int_{\mathbb{D}} |f'(z)|^2 N(\phi, w) dA(w)$$

where $N(\phi, w)$ is the Nevanlinna counting function of ϕ :

$$N(\phi, w) = \sum_{\phi(z)=w} \ln |z|^{-1} \quad \forall w \in \mathbb{D}\backslash\{\phi(0)\}.$$

This formula and the Littlewood-Paley identity show

$$\begin{aligned}
&\|\sigma_b \circ \phi \circ \sigma_a - \sigma_b \circ \phi(a)\|_2^2 \\
= \ & 4 \int_{\mathbb{D}} |(\sigma_b \circ \phi \circ \sigma_a)'(z)|^2 \ln \frac{1}{|z|} dA(z) \\
= \ & 4 \int_{\mathbb{D}} N(\sigma_b \circ \phi \circ \sigma_a, w) \ dA(w)
\end{aligned}$$

Now, if $C_\phi : H^2_{K_1} \to H^2_{K_2}$ is bounded, Then

$$\begin{aligned}
\infty &> \sup_{b\in\mathbb{D}} \|C_\phi f_b\|^2_{H^2_{K_2},*}\\
&= \sup_{b\in\mathbb{D}} \|f_b\circ\phi\|^2_{H^2_{K_2},*}\\
&\simeq \sup_{a,b\in\mathbb{D}} \frac{1-|a|^2}{K_2(1-|a|^2)}\int_{\mathbb{D}} |(f_b\circ\phi\circ\sigma_a)'(z)|^2 \ln\frac{1}{|z|}dA(z)\\
&= \sup_{a,b\in\mathbb{D}} \frac{1-|a|^2}{K_2(1-|a|^2)}\int_{\mathbb{D}} |(f_b'(\phi\circ\sigma_a))(z)|^2|\phi'(\sigma_a)(z)|^2|\sigma_a'(z)|^2 \ln\frac{1}{|z|}dA(z)\\
&= \sup_{a,b\in\mathbb{D}} \frac{1-|a|^2}{K_2(1-|a|^2)}\int_{\mathbb{D}} \frac{(1-|b|^2)K_1(1-|b|^2)}{|1-\overline{b}\cdot\phi\circ\sigma_a(z)|^4} \ln\frac{1}{|z|}dA(\phi\circ\sigma_a(z))\\
&= \sup_{a,b\in\mathbb{D}} \frac{1-|a|^2}{K_2(1-|a|^2)}\cdot\frac{K_1(1-|b|^2)}{1-|b|^2}\int_{\mathbb{D}} \frac{(1-|b|^2)^2}{|1-\overline{b}\cdot\phi\circ\sigma_a(z)|^4} \ln\frac{1}{|z|}dA(\phi\circ\sigma_a(z))\\
&= \sup_{a,b\in\mathbb{D}} \frac{1-|a|^2}{K_2(1-|a|^2)}\cdot\frac{K_1(1-|b|^2)}{1-|b|^2}\int_{\mathbb{D}} |(\sigma_b'(\phi\circ\sigma_a))(z)|^2 \ln\frac{1}{|z|}dA(\phi\circ\sigma_a(z))\\
&= \sup_{a,b\in\mathbb{D}} \frac{1-|a|^2}{K_2(1-|a|^2)}\cdot\frac{K_1(1-|b|^2)}{1-|b|^2}\int_{\mathbb{D}} \ln\frac{1}{|z|}dA(\sigma_b\circ\phi\circ\sigma_a(z))\\
&= \sup_{a,b\in\mathbb{D}} \frac{1-|a|^2}{K_2(1-|a|^2)}\cdot\frac{K_1(1-|b|^2)}{1-|b|^2}\int_{\mathbb{D}} N(\sigma_b\circ\phi\circ\sigma_a, z)dA(z)\\
&= \sup_{a,b\in\mathbb{D}} \frac{1-|a|^2}{K_2(1-|a|^2)}\cdot\frac{K_1(1-|b|^2)}{1-|b|^2}\cdot\|\sigma_b\circ\phi\circ\sigma_a-\sigma_b\circ\phi(a)\|_2^2\\
&\geqslant \sup_{a\in\mathbb{D}} \frac{1-|a|^2}{K_2(1-|a|^2)}\cdot\frac{K_1(1-|\phi(a)|^2)}{1-|\phi(a)|^2}\cdot\|\sigma_{\phi(a)}\circ\phi\circ\sigma_a\|_2^2\,.
\end{aligned}$$

Hence (2) is true. The proof is ended.

Proof of Theorem 1.2: We first assume $\phi(0) = 0$. By Schwarz Lemma for ϕ, we have $|\phi(a)| \le |a|$, $\forall a \in \mathbb{D}$. And by (2) we have

$$\frac{K(t)}{t}$$

is nonincreasing. Then we deduce that

$$\frac{1-|a|^2}{K(1-|a|^2)}\cdot\frac{K(1-|\phi(a)|^2}{1-|\phi(a)|^2} \le 1 \quad \forall a\in\mathbb{D}.$$

Therefore, it follow from (1) and the Littlewood subordination principle that

$$\begin{aligned}
& \|C_\phi f\|^2_{H^2_K,*} \\
= & \sup_{a\in\mathbb{D}} \frac{1-|a|^2}{K(1-|a|^2)} \cdot \|g_a \circ \psi_a\|^2 \\
\leq & \sup_{a\in\mathbb{D}} \frac{1-|a|^2}{K(1-|a|^2)} \cdot \|g_a\|^2 \\
= & \sup_{a\in\mathbb{D}} \frac{1-|a|^2}{K(1-|a|^2)} \|f\circ\sigma_{\phi(a)} - f\circ\phi(a)\|_2^2 \\
\leq & \|f\|^2_{H^2_K,*} \cdot \sup_{a\in\mathbb{D}} \frac{1-|a|^2}{K(1-|a|^2)} \cdot \frac{K(1-|\phi(a)|^2)}{1-|\phi(a)|^2} \\
\leq & \|f\|^2_{H^2_K,*} .
\end{aligned}$$

For the general case, Let

$$\begin{cases} \psi = \sigma_{\phi(0)} \circ \phi; \\ \lambda = \frac{\overline{a}b-1}{1-a\overline{b}}; \\ b = \phi(0); \\ c = \sigma_a(b). \end{cases}$$

Then we have

$$\sigma_b \circ \sigma_a = \lambda\sigma_c, \sigma_b(a) = \lambda c,$$

and

$$\frac{1}{1-|c|^2} = \frac{|1-\overline{a}b|^2}{(1-|a|^2)(1-|b|^2)} \leq \frac{1+|b|}{(1-|a|^2)(1-|b|)} .$$

Thus, we see $tK(\frac{1}{t})$ is nondecreasing for t, which shows that

$$\frac{K(1-|c|^2)}{1-|c|^2} \leq \frac{1+|b|}{(1-|a|^2)(1-|b|)} K\left[\frac{(1-|a|^2)(1-|b|)}{1+|b|}\right].$$

Then, we obtain

$$
\begin{aligned}
& \|C_{\sigma_b} f\|^2_{H^2_K,*} \\
= \ & \|f\circ\sigma_b\|^2_{H^2_K,*} \\
= \ & \sup_{a\in\mathbb{D}} \frac{1-|a|^2}{K(1-|a|^2)} \|f\circ\sigma_b\circ\sigma_a - f\circ\sigma_b(a)\|_2^2 \\
= \ & \sup_{a\in\mathbb{D}} \frac{1-|a|^2}{K(1-|a|^2)} \|f(\lambda\sigma_c) - f(\lambda c)\|_2^2 \\
\leq \ & \|f\|^2_{H^2_K,*} \cdot \sup_{a\in\mathbb{D}} \frac{1-|a|^2}{K(1-|a|^2)} \cdot \frac{K(1-|c|^2)}{1-|c|^2} \\
\leq \ & \|f\|^2_{H^2_K,*} \cdot \sup_{a\in\mathbb{D}} \frac{1-|a|^2}{K(1-|a|^2)} \cdot \frac{1+|b|}{(1-|a|^2)(1-|b|)} K\left[\frac{(1-|a|^2)(1-|b|)}{1+|b|}\right] \\
= \ & \|f\|^2_{H^2_K,*} \cdot \sup_{a\in\mathbb{D}} \frac{1+|b|}{1-|b|} \cdot \frac{K\left[\frac{(1-|a|^2)(1-|b|)}{1+|b|}\right]}{K(1-|a|^2)} \\
\leq \ & \|f\|^2_{H^2_K,*} \frac{1+|b|}{1-|b|} \cdot \varphi_K\left(\frac{1-|b|}{1+|b|}\right).
\end{aligned}
$$

Moreover, $\phi = \sigma_b \circ \psi$ and $\psi(0) = \sigma_{\phi(0)} \circ \phi(0) = 0$, the Littlewood subordination theorem tells us

$$
\begin{aligned}
\|C_\phi f\|^2_{H^2_K,*} &= \|f\circ\phi\|^2_{H^2_K,*} \\
&= \|f\circ\sigma_b\circ\psi\|^2_{H^2_K,*} \\
&\leq \|f\circ\sigma_b\|^2_{H^2_K,*} \\
&\leq \|f\|^2_{H^2_K,*} \frac{1+|b|}{1-|b|} \cdot \varphi_K\left(\frac{1-|b|}{1+|b|}\right).
\end{aligned}
$$

That is,

$$
\|C_\phi f\|^2_{H^2_K,*} \leq \|f\|^2_{H^2_K,*} \frac{1+|\phi(0)|}{1-|\phi(0)|} \cdot \varphi_K\left(\frac{1-|\phi(0)|}{1+|\phi(0)|}\right).
$$

This complete the proof of Theorem 1.2.

Acknowledgments

The author would like to thank the referees for their careful reading and valuable suggestions.

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