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On certain weighted Schur type inequalities

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Original article

Abstract

In this note we give sharp Schur type inequalities for univariate polynomials with convex weights. Our approach will rely on application of two-dimensional Markov type inequalities, and also certain properties of Jacobi polynomials in order to prove sharpness.

Keywords

- Schur type inequality
- Markov type inequality
- real polynomials

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1. Introduction

In all what follows, $\mathbb{R}$ stands for the sets of real numbers, $\mathbb{N}$ denotes the set of all natural numbers $\{1, 2, 3, \dots\}$, $\mathbb{N}_2 = \{l \in \mathbb{N} : l \geq 2\}$. We denote by $\mathcal{P}_n(\mathbb{R}^N)$ the space of all polynomials of N real variables with real coefficients of degree at most n. We write $\mathcal{P}_n$ instead of $\mathcal{P}_n(\mathbb{R}^1)$. Let $L_p(\Omega)$, $1 \leq p < \infty$, be the space of all Lebesgue-measurable functions f on $\Omega \subset \mathbb{R}^m$ such that

$$\|f\|_{L_p(\Omega)} := (\int_\Omega |f(x)|^p \, dx)^{1/p} < \infty \quad \text{if} \quad 1 \leq p < \infty.$$

In approximation theory Schur and Markov type polynomial inequalities constitute an important subject, see e.g. [7, 12]. The classical inequality of Schur states that

$$\|P\|_{[-1,1]} \leq (n+1)\|\sqrt{1-x^2}P\|_{[-1,1]} \qquad (P \in \mathcal{P}_n), \tag{1}$$

where $\|f\|_K := \sup_{x\in K} |f(x)|$. This can be generalized to weights $(1-x^2)^\beta$ with $\beta > 1/2$ as well (see [1], Lemma 2.4, p. 73):

$$\|P\|_{[-1,1]} \leq C(\beta)n^{2\beta}\|(1-x^2)^\beta P\|_{[-1,1]} \qquad (P \in \mathcal{P}_n).$$

The classical Markov inequality for univariate algebraic polynomials of degree n gives the following sharp upper bounds for their derivatives:

$$\|P'\|_{[a,b]} \leq \frac{2n^2}{b-a}\|P\|_{[a,b]}. \tag{2}$$

There are many generalizations and variations of the classical inequality of Schur and the classical Markov inequality, see recent work in [2, 4, 5, 6, 9, 10, 11, 13] and [14]. In order to verify our main result we shall need the following generalization of the classical Markov inequality.

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Theorem 1.1 *Let $f:(-1,0]\to(0,\infty)$ be a $C^2((-1,0))$ function and there exists $-1<\eta\le 0$ such that $f|_{(-1,\eta)}$ is convex function so that*

$$\lim_{x\to -1^+} f'(x) = \lim_{x\to -1^+} f(x) = 0.$$

Suppose that there exists a constant $k\in\mathbb{N}_2$ such that $(f)^{1/k}$ is a concave function on the interval $(-1,\eta)$. Let

$$K=\{(x,y)\in\mathbb{R}^2 : -1<x<1,\ 0\le y\le f(-|x|)\}.$$

Then, for every $1\le p<\infty$, there exists a constant $C>0$ such that

$$\left\|\frac{\partial P}{\partial y}\right\|_{L_p(K)} \le C\frac{n^2}{f'(-1+1/n^2)}\|P\|_{L_p(K)}. \tag{3}$$

for $n\in\mathbb{N}_2$ and every polynomial $P\in\mathcal{P}_n(\mathbb{R}^2)$.

The above theorem can be proved by using techniques similar to those used in [3]. Another way, the inequality (3) can be proved using the main result of [11]. This is how we get

$$\left\|\frac{\partial P}{\partial y}\right\|_{L_p(K)} \le \hat{C}\frac{1}{f(-1+1/n^2)}\|P\|_{L_p(K)}.$$

Then using the properties of the function f it can be shown that

$$\frac{1}{f(-1+1/n^2)} \le k\frac{n^2}{f'(-1+1/n^2)}.$$

Our goal is to establish a certain generalization of (1) by using (3).

2. Main results

This section addresses main theorems.

Theorem 2.1 *Let $f:(-1,0]\to(0,\infty)$ be a $C^2((-1,0))$ function and there exists $-1<\eta\le 0$ such that $f|_{(-1,\eta)}$ is convex function so that*

$$\lim_{x\to -1^+} f'(x) = \lim_{x\to -1^+} f(x) = 0.$$

Suppose that there exists a constant $k\in\mathbb{N}_2$ such that $(f)^{1/k}$ is a concave function on the interval $(-1,\eta)$. Let

$$w(x)=\begin{cases} f(-|x|) & \text{if } x\in(-1,1)\\ 0 & \text{otherwise.}\end{cases}$$

Then, for every $1\le p<\infty$, there exists a constant $B>0$ such that

$$\left\|w^{1/p}P\right\|_{L_p([-1,1])} \le B\frac{n^2}{f'(-1+1/n^2)}\|w^{1/p+1}P\|_{L_p([-1,1])}. \tag{4}$$

for $n\in\mathbb{N}_2$ and every polynomial $P\in\mathcal{P}_n$.

Proof. Let $P \in \mathcal{P}_n$ and consider the polynomial $Q(x,y) = yP(x)$. It is clear that $\frac{\partial Q}{\partial y} = P$. Let $K = \{(x,y) \in \mathbb{R}^2 : -1 < x < 1,\, 0 \le y \le f(-|x|)\}$. Then, by (3), we have

$$\left\| \frac{\partial Q}{\partial y} \right\|_{L_p(K)} \le C \frac{(n+1)^2}{f'(-1+1/(n+1)^2)} \|Q\|_{L_p(K)}.$$

Hence

$$\int_{-1}^{1}\int_{0}^{w(x)} |P(x)|^p \, dydx \le \left(\frac{C(n+1)^2}{f'(-1+1/(n+1)^2)} \right)^p \int_{-1}^{1}\int_{0}^{w(x)} |yP(x)|^p \, dydx.$$

Therefore

$$\int_{-1}^{1} w(x)\, |P(x)|^p \, dx \le \left(\frac{C(n+1)^2}{f'(-1+1/(n+1)^2)} \right)^p \int_{-1}^{1} \frac{w(x)}{p+1} \, |w(x)P(x)|^p \, dx.$$

Now we wish to prove that there exists a positive constant D such that

$$C \frac{(n+1)^2}{f'(-1+1/(n+1)^2)} \le D \frac{n^2}{f'(-1+1/n^2)}. \tag{5}$$

If $\frac{1}{(n+1)^2} - 1 < \eta$ then, by the assumptions

$$\lim_{x \to -1^+} f'(x) = \lim_{x \to -1^+} f(x) = 0$$

and $(f)^{1/k}$ is a concave function on the interval $(-1, \eta)$,

$$\frac{1}{n^2} f'(-1+1/n^2) \le k f(-1+1/n^2),$$

$$\left(\frac{n+1}{n} \right)^{2k} f(-1+1/(n+1)^2) \ge f(-1+1/n^2).$$

By the fact that f is convex, we have

$$f'(-1+1/(n+1)^2) \ge (1+n)^2 f(-1+1/(1+n)^2).$$

Thus,

$$k4^k \frac{f'(-1+1/(n+1)^2)}{(1+n)^2} \ge \frac{f'(-1+1/n^2)}{n^2}. \tag{6}$$

It now follows from (6) that (5) holds with $D = Ck4^k$, which completes the proof.

The next theorem shows that the inequality (4) is asymptotically sharp.

Theorem 2.2 *Let f, k and w be as in Theorem 2.1. Then, for every $1 \le p < \infty$ and $n \in \mathbb{N}_2$, there exist a positive constant $B_1 > 0$ and a sequence of polynomials $U_n \in \mathcal{P}_n$ such that*

$$\left\| w^{1/p} U_n \right\|_{L_p([-1,1])} \ge B_1 \frac{n^2}{f'(-1+1/n^2)} \|w^{1/p+1} U_n\|_{L_p([-1,1])}. \tag{7}$$

Proof. Let $U_n(x) = P_n^{(\omega,\omega)}(-x)$. Here $P_n^{(\omega,\sigma)}$ denotes the Jacobi polynomial of degree n associated with parameters ω, σ. Then

$$\left\|w^{1/p}U_n\right\|_{L_p([-1,1])}^p \geq \int_{-1}^{-1+\frac{1}{n^2}} f(x)|P_n^{(\omega,\omega)}(-x)|^p\,dx. \tag{8}$$

With elementary changes of variables, we arrive at the following equation

$$\int_{-1}^{-1+\frac{1}{n^2}} f(x)|P_n^{(\omega,\omega)}(-x)|^p\,dx = \int_0^{\frac{1}{n^2}} f(t-1)|P_n^{(\omega,\omega)}(1-t)|^p\,dt. \tag{9}$$

By making the change of variable $t = \frac{z}{2n^2}$, we obtain

$$\int_0^{\frac{1}{n^2}} f(t-1)|P_n^{(\omega,\omega)}(1-t)|^p\,dt = \frac{1}{2n^2}\int_0^2 f(\tau_n(z)-1)|P_n^{(\omega,\omega)}(1-\tau_n(z))|^p\,dz, \tag{10}$$

where $\tau_n(z) = \frac{z}{2n^2}$. By the formula of Mehler-Heine type (see [15], Theorem 8.1.1.)

$$\frac{1}{2n^2}|P_n^{(\omega,\omega)}(1-\tau_n(z))|^p \geq \frac{n^{\omega p}}{4^p n^2}(4(z/2)^{-\omega}J_\omega(z) - 1/\Gamma(\omega+2))^p$$

for $\omega > 0$, and all sufficiently large n. Here $J_\omega(z)$ is the Bessel functions of the first kind. Let $g_n(z) = \tau_n(z) - 1$. Since

$$\min_{z\in[0,2]}\{(z/2)^{-\omega}J_\omega(z)\} \geq \min_{z\in[0,2]}\left\{\frac{1}{\Gamma(\omega+1)} - \frac{z^2}{4\Gamma(\omega+2)}\right\} = \frac{\omega}{\Gamma(\omega+2)},$$

we have

$$\frac{1}{2n^2}\int_0^2 f(g_n(z))|P_n^{(\omega,\omega)}(-g_n(z))|^p\,dz \geq \left(\frac{4\omega-1}{4\Gamma(\omega+2)}\right)^p n^{\omega p-2}\int_0^2 f(g_n(z))\,dz. \tag{11}$$

Then integration by parts shows that

$$\int_0^2 f(g_n(z))\,dz = 2f(-1+1/n^2) - \frac{1}{2n^2}\int_0^2 zf'(g_n(z))\,dz.$$

If $x \in (-1,\eta)$ then $kf(x) \geq (1+x)f'(x)$. This leads to

$$\int_0^2 f(g_n(z))\,dz \geq \frac{2f(-1+1/n^2)}{k+1} \geq \frac{2f'(-1+1/n^2)}{n^2k(k+1)}. \tag{12}$$

From (8)-(12) we see that

$$\left\|w^{1/p}U_n\right\|_{L_p([-1,1])}^p \geq \left(\frac{4\omega-1}{4\Gamma(\omega+2)}\right)^p \frac{2n^{\omega p-4}f'(-1+1/n^2)}{k(k+1)}. \tag{13}$$

On the other hand, by the assumption on f, there exists $M > 0$ such that

$$\left\|w^{1/p+1}U_n\right\|_{L_p([\eta,-\eta])}^p \leq M^{p+1}\int_\eta^{-\eta} |P_n^{(\omega,\omega)}(-x)|^p\,dx. \tag{14}$$

Applying certain properties of Jacobi polynomials $P_n^{(\omega,\sigma)}$ verified in [15], (7.32.5), p. 169, one can show that there exists $\Lambda > 0$ such that

$$\int_{\eta}^{-\eta} |P_n^{(\omega,\omega)}(-x)|^p \, dx \le \Lambda n^{-p/2} \tag{15}$$

If $-1 < -1 + 1/n^2 < \eta$ then $\frac{1}{n^2} f'(-1+1/n^2) \ge f(-1+1/n^2)$. By the fact that $(f)^{1/k}$ is concave on the interval $(-1,\eta)$, there exists an absolute constant $\Lambda_1 > 0$ such that

$$f(-1+1/n^2) \ge \frac{\Lambda_1}{n^{2k}}.$$

Hence, by (15) and (14), there exists constant $\Lambda_2 > 0$ such that

$$\left\| w^{1/p+1} U_n \right\|_{L_p([\eta,-\eta])}^p \le \Lambda_2 \left(f'(-1+1/n^2) \right)^{p+1} n^{\omega p - 4 - 2p}. \tag{16}$$

for sufficiently large ω. By the definition of w and the symmetry relation

$$P_n^{(\omega,\sigma)}(-z) = (-1)^n P_n^{(\sigma,\omega)}(z),$$

we have

$$\left\| w^{1/p+1} U_n \right\|_{L_p([-1,-1+1/n^2])} = \left\| w^{1/p+1} U_n \right\|_{L_p([1-1/n^2,1])}. \tag{17}$$

Therefore, it is enough to consider the norm on one of these intervals. Since $f|_{(-1,\eta)}$ is convex,

$$\left\| w^{1/p+1} U_n \right\|_{L_p([-1,-1+1/n^2])}^p \le (f(-1+1/n^2))^{p+1} \int_{-1}^{-1+\frac{1}{n^2}} |P_n^{(\omega,\omega)}(-x)|^p \, dx.$$

Hence

$$\left\| w^{1/p+1} U_n \right\|_{L_p([-1,-1+1/n^2])}^p \le (f(-1+1/n^2))^{p+1} \int_{-1}^{\eta} |P_n^{(\omega,\omega)}(-x)|^p \, dx.$$

By Theorem 4.5 of [8], there is an absolute constant $D > 0$ such that

$$\int_{-1}^{\eta} |P_n^{(\omega,\omega)}(-x)|^p \, dx \le D \int_{-1+1/n^2}^{\eta} |P_n^{(\omega,\omega)}(-x)|^p \, dx.$$

Thus

$$\left\| w^{1/p+1} U_n \right\|_{L_p([-1,-1+1/n^2])}^p \le D (f(-1+1/n^2))^{p+1} \int_{-1+1/n^2}^{\eta} |P_n^{(\omega,\omega)}(-x)|^p \, dx. \tag{18}$$

Let $\rho := \arccos(-\eta)$ and $\lambda_n := \arccos(1 - 1/n^2)$. Using the change of variables $x = -\cos\theta$, we have

$$\int_{-1+1/n^2}^{\eta} |P_n^{(\omega,\omega)}(-x)|^p \, dx = \int_{\lambda_n}^{\rho} |P_n^{(\omega,\omega)}(\cos\theta)|^p \sin\theta \, d\theta. \tag{19}$$

Since $\sin x \leq x$ for $x \geq 0$, we have

$$\int_{\lambda_n}^{\rho} |P_n^{(\omega,\omega)}(\cos\theta)|^p \sin\theta \, d\theta \leq \int_{\lambda_n}^{\rho} |P_n^{(\omega,\omega)}(\cos\theta)|^p \theta \, d\theta. \tag{20}$$

Applying certain properties of Jacobi polynomials $P_n^{(\omega,\sigma)}$ verified in [15], (7.32.5), p. 169, again, we conclude that there exists a natural number n_1 so that

$$\int_{\lambda_n}^{\rho} |P_n^{(\omega,\omega)}(\cos\theta)|^p \theta \, d\theta \leq D_1 n^{-p/2} \int_{\lambda_n}^{\rho} \theta^{-p(\omega+1/2)+1} d\theta.$$

for $n \geq n_1$ and appropriately adjusted constant D_1. Therefore,

$$\int_{\lambda_n}^{\rho} |P_n^{(\omega,\omega)}(\cos\theta)|^p \theta \, d\theta \leq D_1 n^{-p/2} \frac{\lambda_n^{-p(\omega+1/2)+2}}{p(\omega+1/2)-2}$$

for sufficiently large ω. One can verify easily that there exists a positive constant D_2 so that, for $n \geq 2$, $\frac{1}{n} \leq D_2 \lambda_n$. Hence,

$$\int_{\lambda_n}^{\rho} |P_n^{(\omega,\omega)}(\cos\theta)|^p \theta \, d\theta \leq \frac{D_1}{p(\omega+1/2)-2} (D_2 n)^{\omega p - 2}. \tag{21}$$

By (18)-(21), there is a constant $D_3 > 0$ such that

$$\left\| w^{1/p+1} U_n \right\|^p_{L_p([-1,-1+1/n^2])} \leq D_3 (f(-1+1/n^2))^{p+1} n^{\omega p - 2}.$$

Using the assumption that $f|_{(-1,\eta)}$ is convex again yields

$$(f(-1+1/n^2))^{p+1} n^{p\omega - 2} \leq (f'(-1+1/n^2))^{p+1} n^{\omega p - 2p - 4}.$$

Thus

$$\left\| w^{1/p+1} U_n \right\|^p_{L_p([-1,-1+1/n^2])} \leq D_3 (f'(-1+1/n^2))^{p+1} n^{\omega p - 2p - 4}. \tag{22}$$

Combining (16), (17) and (22) leads to

$$\left\| w^{1/p+1} U_n \right\|^p_{L_p([-1,1])} \leq (\Lambda_2 + 2D_3)(f'(-1+1/n^2))^{p+1} n^{\omega p - 2p - 4}. \tag{23}$$

The inequalities (23) and (13) give the desired result.

3. Concluding remarks

At the end of this article, we would like to make some comments related to the techniques used in this work.

- Let $k \in \mathbb{N}$, $k \geq 2$. If $1 < r \leq k$, then we can apply the main results (Theorem 2.1 and Theorem 2.2) to the following functions:
  - $f_1(x) = b_1(1+x)^r$,
  - $f_2(x) = (1+x)^r \ln(-\ln(b_2(1+x)))$,

* $f_3(x) = -(1+x)^r \ln(b_3(1+x))$,
* $f_4(x) = (1+x)^r(-\ln(b_4(1+x)))^c$,
* $f_5(x) = (1-x^2)^r$,

considered on $(-1,0]$ for appropriately adjusted constants b_1, b_2, b_3, b_4 and c.

- The techniques used in this paper can be applied to the more general functions. Using the higher-dimensional equivalent of Theorem 1.1, we can infer a corresponding generalization of the Schur type inequality.
- The work is devoted to univariate polynomials. Nevertheless, in many cases the problem of finding the analog of inequality (3) for higher dimensional domains can be reduced to a two dimensional situation. Here are some examples:

* $\{(x,y) \in \mathbb{R}^2 : 0 \le x \le 1, \quad ax^k \le y \le x^r\} \times [0,1]^m$,
* $\{(x,y) \in \mathbb{R}^{m+1} : |x| \le 1, \quad a|x|^k \le y \le |x|^r\}$,
* $\{x \in \mathbb{R}^m : a \le |x_1|^k + |x_2|^k + \ldots + |x_m|^k, \quad |x_1|^r + |x_2|^r + \ldots + |x_m|^r < 1\}$,
* $\{(x,y,z) \in \mathbb{R}^3 : 0 \le x \le 1, ax^k \le y \le x^r, Ax + By + C_1 \le z \le Ax + By + C_2\}$,

for every $k, m \in \mathbb{N}$, $r, A, B, C_1, C_2 \in \mathbb{R}$ so that $k \ge 2$, $r \le k$, $0 < a < 1$, $C_1 < C_2$ and $Ax + By + C_1 \ge 0$ if $0 \le x \le 1$, $ax^k \le y \le x^r$. Here $|x| := \sqrt{x_1^2 + x_2^2 + \ldots + x_m^2}$.
This then we can use to prove an analogs of inequality (4).

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