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Polynomial approximation of regular functions of a quaternionic variable

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Original article

Abstract

We consider Bernstein-Walsh-Siciak-type theorems on the polynomial approximation in the case of regular functions of one quaternionic variable and their applications to the uniform approximation and approximation in L^p norms with respect to measures satisfying the Bernstein-Markov condition.

Keywords

- quaternionic regular functions
- polynomial approximation
- Bernstein-Walsh-Siciak theorem
- Bernstein-Markov condition
- polynomial extremal function
- Bernstein-Walsh inequality

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1. Introduction

The analytic continuation of a function $f : \mathbb{C}^n \supset E \mapsto \mathbb{C}$ and the rate of convergence of polynomials approximating function f in the uniform norm on a compact set E was studied in the case of one and several complex variables by Bernstein, Walsh, Siciak and other autors (see [8] and references given there). In this paper we prove analogues of the Bernstein-Walsh-Siciak theorem in the case of regular function $f : \mathbb{H} \supset E \mapsto \mathbb{H}$ of one quaternionic variable, see Theorems 5.1 and 6.1. Next we consider also relationship between leading terms $\widehat{t}_n(q) = q^n a_n$ of polynomials $t_n(q) = a_0 + qa_1 + \cdots + q^n a_n$ approximating function $f : \mathbb{H} \supset E \mapsto \mathbb{H}$ and regularity of f in an open neighbourhood of the set E, see Propositions 7.1 and 7.2. We also propose certain sufficient conditions for regularity of f expressed by the distribution of points q_{nk} appearing in the factorization of the polynomials of the best approximation

$$t_n(q) = (q - q_{n1}) * (q - q_{n2}) * \cdots * (q - q_{nn})a_{nn},$$

see Propositions 8.1, 8.4 and 8.5.

2. Preliminaries

Let $\mathbb{H}$ be the field of real quaternions with elements $q = x_0 + ix_1 + jx_2 + kx_3$, where the numbers x_0, x_1, x_2, x_3 are real, and i, j, k are *imaginary units*, i.e. their square equals -1 and $ij = -ji = k$, $jk = -kj = i$ and $ki = -ik = j$. We donote by $\Re q := x_0$ the *scalar* (or *real*) part and by $\Im q := ix_1 + jx_2 + kx_3$ the *vector* (or *imaginary*) part of the quaternion q. Let

$$\mathbb{S} := \{q = x_0 + ix_1 + jx_2 + kx_3 \in \mathbb{H} : \ x_0 = 0, \ x_1^2 + x_2^2 + x_3^2 = 1\}$$

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be the unit sphere of purely imaginary quaternions. The elements $I \in \mathbb{S}$ are called *imaginary units* as $I^2 = -1$. In particular, $i = (0,1,0,0)$, $j = (0,0,1,0)$, $k = (0,0,0,1) \in \mathbb{S}$.

One may prove that $\overline{q_1 q_2} = \overline{q_2}\,\overline{q_1}$, where $\overline{q} := x_0 - ix_1 - jx_2 - kx_3 = \Re q - \Im q$ is the conjugate of the element $q = x_0 + ix_1 + jx_2 + kx_3 = \Re q + \Im q$. Then $q\overline{q} = \overline{q}q = x_0^2 + x_1^2 + x_2^2 + x_3^2$. Observe that $|q| := (q\overline{q})^{1/2}$ is a norm of the quaternion q. We have also $q = |q|(\cos\varphi + \hat{q}\sin\varphi)$ for an angle $\varphi \in [0,\pi]$ where $\hat{q}$ is *the axis of* q defined by $\hat{q} := \frac{1}{|q|\sin\varphi}(ix_1 + jx_2 + kx_3) = \frac{\Im q}{|q|\sin\varphi}$ if $|\Im q|^2 = x_1^2 + x_2^2 + x_3^2 \neq 0$, or $\hat{q} := i\,0 + j\,0 + k\,0$, otherwise.

We consider the set $\mathbb{H}_d[q] := \{\sum_{n=0}^{d} q^n a_n, a_n \in \mathbb{H}\}$ of *regular polynomials* of one quaternionic variable q of degree less or equal d with coefficients on the right side of the monomials q^n and the set $\mathbb{H}[q] = \bigcup_{d=0}^{\infty} H_d[q]$ of all regular polynomials of one quaternionic variable $q \in \mathbb{H}$. Following [4] we define *the regular product* $*$ of polynomials $f(q) = \sum_{k=0}^{m} q^k a_k$ and $g(q) = \sum_{l=0}^{n} q^l b_l$:

$$f * g(q) := \sum_{p=0}^{mn} q^p c_p, \text{ where } c_p = \sum_{k+l=p} a_k b_l. \tag{1}$$

Observe that the regular product $p_1 * p_2$ of elements $p_1 \in \mathbb{H}_k[q]$, $p_2 \in \mathbb{H}_l[q]$ is an element of $\mathbb{H}_{k+l}[q]$ while the simple product $p_1\, p_2$ of the factors p_1 and p_2 need not be an element of the set $\mathbb{H}[q]$:

$$\begin{aligned} (q-j)*(q-k) &= q^2 - q(j+k) + jk &&\in \mathbb{H}[q],\\ (q-j)\,(q-k) &= q^2 - jq - qk + jk &&\notin \mathbb{H}[q], \end{aligned}$$

which is due to noncommutability of quaternions.

By the Eilenberg-Niven theorem (known also as the *quaternionic version of the Fundamental Theorem of Algebra*) one may factor each polynomial in a quaternionic variable. In particular, for each regular quaternionic polynomial

$$f(q) = a_1 + qa_1 + q^2 a_2 + \cdots + q^n a_n \in \mathbb{H}_n[q]$$

there are quaternions $q_1, q_2, \ldots, q_n$ such that

$$f(q) = (q - q_1) * (q - q_2) * \cdots * (q - q_n)a_n \tag{2}$$

(see [5], Theorem 3.18, Corollary 3.19).

In the following discussion, we will call the elements of the set $\mathbb{H}[q]$ just *polynomials*, omitting the word *regular*.

Observe that the factorisation (2) need not be unique nor the numbers q_2, $q_3, \ldots, q_n$ do not have to be zeroes of the factored polynomial. Let us recall two known examples (see eg. [4]).

Example 2.1 The polynomial $p(q) = (q - i) * (q - 2j)$ can also be factored as

$$p(q) = \left(q - \frac{8i + 6j}{5}\right) * \left(q - \frac{4j - 3i}{5}\right).$$

Example 2.2 Let $p(q) = (q - j) * (q - k) = q^2 - q(j + k) + jk$. We have

$$p(j) = j^2 - j(j + k) + jk = 0$$
$$p(k) = k^2 - k(j + k) + jk = 2jk \neq 0.$$

Even if the rest of the points $q_2,\ q_3,\ \ldots q_n$ in factorisation (2) are not the zeroes of the polynomial, they are related to them (see [7]). Namely, consider the equivalence class of the the quaternion $q_0 \in \mathbb{H}$

$$[q_0] := \{q \in \mathbb{H} :\ \text{there exists}\ a \in \mathbb{H} : a^{-1}qa = q_0\}.$$

One may prove the following remark (see [7], Proposition 4).

Remark 2.3 If $f(q) = (q - q_1) * (q - q_2) * \cdots * (q - q_n)$ then

$$\text{Zero}(f) \subset [q_1] \cup [q_2] \cup \cdots \cup [q_n],$$

where $\text{Zero}(f) = \{q \in \mathbb{H} : f(q) = 0\}$.

3. Regular functions

An extensive survey of the theory of regular functions of one quaternionic variable is presented in [5] and we restrict ourselves to the necessary definitions and properties of regular functions. For $I \in \mathbb{S}$ we donote by L_I the complex plane passing through the origin and containing 1 and I, i.e. $L_I := \mathbb{R} + I\mathbb{R}$. Following [5] we say that a domain $\Omega \subset \mathbb{H}$ that intersects the real axis is called *a slice domain* if, for all imaginary units $I \in \mathbb{S}$, the intersection $\Omega_I := \Omega \cap L_I$ with the complex plane L_I is a domain of L_I. A real differentiable function $f : \Omega \mapsto \mathbb{H}$, defined on a slice domain $\Omega \subset \mathbb{H}$, is called *regular* if for every $I \in \mathbb{S}$ its restriction f_I to the complex line L_I is holomorphic on Ω_I (see [5], Definition 1.1), i.e.

$$\bar{\partial}_I f(x + yI) := \frac{1}{2}\left(\frac{\partial}{\partial x} + I\frac{\partial}{\partial y}\right) f_I(x + yI) \equiv 0 \text{ on } \Omega_I.$$

It is known (see [3]) that the monomial $q^n a$ with $a \in \mathbb{H}$ is regular as well as the sum of regular functions is regular.

A set $T \subset \mathbb{H}$ is called *symmetric* if for all points $x + Iy \in T$, with $x, y \in \mathbb{R}$ and $I \in \mathbb{S}$, the set T contains the whole sphere $x + y\mathbb{S}$, see [5], Definition 1.14. Symmetric slice domains play an important role in the theory of regular functions. We recall the following lemma (see [5], Lemma 1.22):

Lemma 3.1 *Let $\Omega \subset \mathbb{H}$ be a symmetric slice domain and let $I \in \mathbb{S}$. If $f_I : \Omega_I \mapsto \mathbb{H}$ is holomorphic then there exists a unique regular function $g : \Omega \mapsto \mathbb{H}$ such that $g_I = f_I$ in Ω_I.*

As a consequence one may obatain *the extension theorem for regular functions*, see [5], Theorem 1.24.

Theorem 3.2 *Let f be a regular function on a slice domain Ω. There exists a unique regular function $\widetilde{f} : \widetilde{Q} \mapsto \mathbb{H}$ that extends f to the symmetric completion of Ω, i.e. to the set*

$$\widetilde{\Omega} := \bigcup_{x+yI \in \Omega} (x + y\mathbb{S}).$$

Consider the set of power series

$$\sum_{n=0}^{\infty} q^n a_n, \ a_n \in \mathbb{H}. \tag{3}$$

endowed with the natural uniform convegence on compact sets. Observe that if $R := (\limsup_{n\to\infty} \sqrt[n]{|a_n|})^{-1}$ then the series (3) converges uniformly on compact subsets of $B(0, R) := \{q \in \mathbb{H} : |q| < R\}$ to a regular function $f(q) = \sum_{n=0}^{\infty} q^n a_n$ in $B(0, R)$ and diverges if $|q| > R$, see [5], Theorem 1.6. We define *the regular product* $*$ of the series $f(q) = \sum_k q^k a_k$ and $g(q) = \sum_l q^l a_l$ similarily as the regular product of polynomials (1):

$$f * g(q) := \sum_m q^m \sum_{k+l=m} a_k b_l \tag{4}$$

The regular product $f * g$ is regular in $B(0, R)$ if f, g are regular in $B(0, R)$. One may also prove the proposition (see [5], Proposition 1.28).

Proposition 3.3 *The set of regular functions on a symmetric slice domain $\Omega \subset \mathbb{H}$ is a noncommutative ring with respect to $+$ and $*$. In particular polynomials $\mathbb{H}[q]$ are regular in $\mathbb{H}$.*

One may define a distance $\sigma : \mathbb{H} \times \mathbb{H} \mapsto \mathbb{R}$ (see [5])

$$\sigma(p, q) = \begin{cases} |p - q|, & \text{if } p, q \text{ lie on the same complex line } L_I \\ \omega(p, q), & \text{otherwise} \end{cases} \tag{5}$$

where

$$\omega(p, q) = \sqrt{(\Re p - \Re q)^2 + (|\Im p| + |\Im q|)^2}.$$

The topology τ_σ in $\mathbb{H}$ defined by the distance (5) is finer that the Euclidean topology τ_d induced by the distance $d(p, q) = |p - q|$, see [5], Section 2.13. Let $\Sigma(c, R) := \{q \in \mathbb{H} : \sigma(c, q) < R\}$ be the σ ball centered at $c \in \mathbb{H}$ of radii $R > 0$ (see Figures 1, 2, 3).

We say (see [5], Definition 2.13) that $f : \mathbb{H} \supset \Omega \mapsto \mathbb{H}$ is *σ-analytic at $c \in \Omega$* if there exists $R > 0$ and a regular power series $\sum_{n=0}^{\infty} (q - c)^{*n} a_n$, where

$$(q - c)^{*n} a_n := \underbrace{(q - c) * (q - c) * \cdots * (q - c)}_{\text{regular product of } n \text{ factors } q-c} a_n,$$

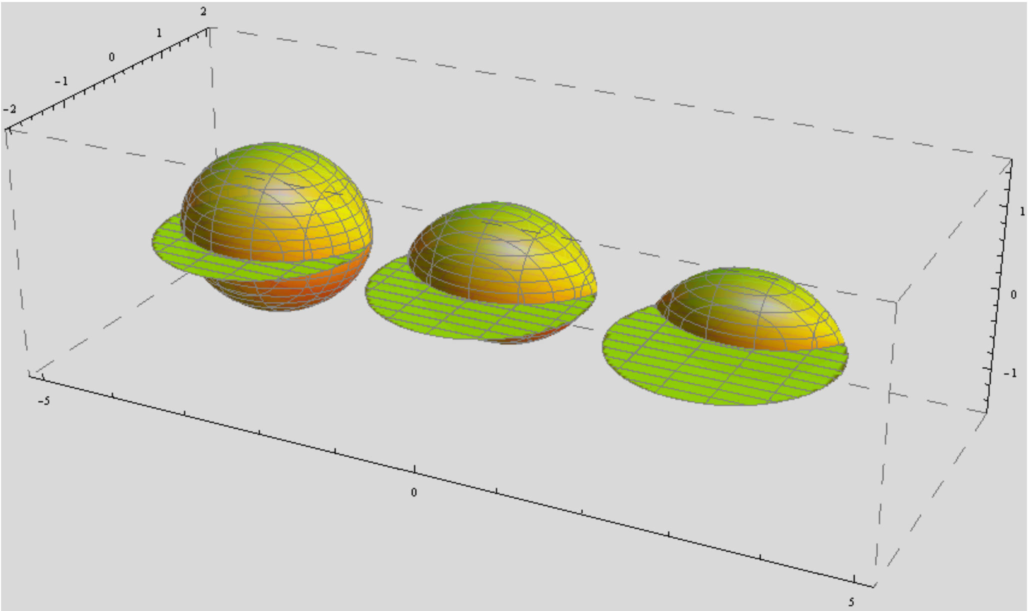

Figure 1: The sets $\Sigma(c_i, R_i) \cap \{x_0 + ix_1 + jx_2 + kx_3 \in \mathbb{H} : x_3 = 0\}$ with $c_1 = (-3, -\frac{3}{10}, 0, 0)$, $c_2 = (0, -\frac{5}{10}, 0, 0)$, $c_3 = (3, -\frac{7}{10}, 0, 0)$ and $R_1 = R_2 = R_3 = \frac{13}{10}$.

such that $f(q) = \sum_{n=0}^{\infty}(q - c)^{*n}a_n$ for $q \in \Sigma(c, R)$. We say that f is *σ-analytic in* Ω if it is σ-analytic at all $c \in \Omega$.

Regularity and $\sigma-$analyticity are strictly related, see [5], Corollary 2.14.

Proposition 3.4 *A quaternionic function is regular in a domain if and only if it is σ-analytic in the same domain.*

We recall *the Splitting Lemma* that is a crucial tool in the theory of regular functions, see [5], Lemma 1.

Lemma 3.5 *Let f be a regular function defined on an open set Ω. Then for any $I \in \mathbb{S}$ and any $J \in \mathbb{S}$ with $J \perp I$, there exist two holomorphic functions $F, G : \Omega_I \cap L_I \mapsto L_I$ such that for every $z = x + yI$ we have $f_I(z) = F(z) + G(z)J$.*

As a consequence we obtain

Corrolary 3.6 *Let $\Omega \subset \mathbb{H}$ be a symmetric slice domain and let $p_n \in \mathbb{H}_n[q]$ be a sequence of polynomials converging uniformly on compact subsets of Ω to a function f bounded on compact subsets of Ω. Then f is regular on Ω.*

Proof. Let $K \subset \Omega$ be a compact. Fix $I, J \in \mathbb{S}$, $I \perp J$. For $n = 1, 2, 3, \ldots$ by the Splitting Lemma we get holomorphic functions $F_n : \Omega \cap L_I \mapsto L_I$, $G_n : \Omega \cap L_I \mapsto L_I$, such that $p_{nI}(z) = F_n(z) + G_n(z)J$, and F_n, G_n are converging on compact sets $K \cap L_I$ to functions $F_I : \Omega \cap L_I \mapsto L_I$, $G_I : \Omega \cap L_I \mapsto L_I$ holomorphic on

$\Omega \cap L_I$. Hence $f_I(z) = F_I(z) + G_I(a)J$ for $z \in L_I$ with holomorphic functions $F_I, G_I : \Omega \cap L_I \mapsto L_I$.

4. Polynomial extremal function

Following [8] we define *polynomial extremal function* of a non-empty compact set $E \subset \mathbb{H}$ (called *the Leja-Siciak polynomial extremal function* in the case where $E \subset \mathbb{C}^N$) by

$$\Phi_E(q) := \sup\left\{|p(q)|^{1/\deg p}, p \in \mathbb{H}[q], ||p||_E \leq 1\right\} \tag{6}$$

where

$$\|p\|_E = \sup\{|p(q)|, q \in E\}$$

is the supremum norm of the polynomial $p(q) = a_0 + qa_1 + q^2a_2 + \cdots + q^n a_n \in \mathbb{H}[q]$ on the set E.

Consider the polynomials $p_n(q) = \sum_{k=0}^{n}(q-c)^{*k}a_k$ satisfying $\|p_n\|_E \leq 1$ on the closure of the σ-ball $\Sigma(c, R)$, i.e. on the set $E := \overline{\Sigma(c, R)} = \{q \in \mathbb{H} : \sigma(q, c) \leq R\}$. One may prove that

$$|(q-c)^{*n}| = |(q-c) * (q-c) * \cdots * (q-c)| \leq \sigma(q, c)^n$$

and $\lim_{n\to\infty} |(q-c)^{*n}|^{1/n} = \sigma(q, c)$, see [5], Proposition 2.10. Hence we get the explicit formula for the polynomial extremal function of the set E.

Proposition 4.1 *If $E = \{q \in \mathbb{H} : \sigma(q, c) \leq R\}$ then*

$$\Phi_E(q) := \max\left\{1, \frac{\sigma(q, c)}{R}\right\}.$$

In particular, for $c = 0$ we have $\sigma(q, 0) = |q|$ and we get $\Phi_E(q) := \max\{1, \frac{|q|}{R}\}$.

Observe that function Φ_E need not be continuous. For a finite set $E := \{a_1, a_2, \ldots, a_m\} \subset \mathbb{H}$, we have

$$\Phi_E(q) = \begin{cases} 1, & q \in E, \\ \infty, & q \notin E. \end{cases}$$

We shall say that non-empty compact set $E \subset \mathbb{H}$ is *L-regular* if the polynomial extremal function Φ_E is continuous on $\mathbb{H}$.

By the definition (6) we obtain *the Bernstein-Walsh inequality* (see [8] in the case where $E \subset \mathbb{C}^n$).

Proposition 4.2 *Let $p_n(q) = a_0 + qa_1 + q^2a_2 + \cdots + q^n a_n \in \mathbb{H}_n[q]$ and $E \subset \mathbb{H}$ be a non-empty compact set. Then for any quaternion $q \in \mathbb{H}$ we have*

$$|p_n(q)| \leq \Phi_E^n(q)\|p_n\|_E. \tag{7}$$

In particular, we have

$$|p_n(q)| \leq R^n ||p_n||_E \text{ for } q \in E_R$$

where $E_R := \{q \in \mathbb{H} : \Phi_E(q) \leq R\}$ *for* $R \geq 1$ *is a sublevel set of the function* Φ_E.

5. Bernstein-Walsh-Siciak theorem

Let us propose a version of the Bernstein-Walsh-Siciak theorem for function $f : \mathbb{H} \supset E \mapsto \mathbb{H}$ (see [8], Section 10, Theorem 1 for the case where $f : \mathbb{C}^n \supset E \mapsto \mathbb{C}$).

Theorem 5.1 *Let* $E \subset \mathbb{H}$ *be a non-empty compact L-regular set and let* $f : E \mapsto \mathbb{H}$ *be a bounded function. Let* $p_n \in \mathbb{H}_n[q]$ *be a sequence of polynomials. If there exists* $R > 1$ *such that*

$$\limsup_{n\to\infty} ||f - p_n||_E^{1/n} \leq \frac{1}{R}, \text{ for } R > 1 \tag{8}$$

then

1. *the sequence* p_n *converges uniformly in* $E_r := \{q \in \mathbb{H} : |\Phi_E(q)| \leq r\}$ *for* $1 < r < R$,
2. *function* f *is regular in the interior of the set* E_R, *i.e. there exists regular function* $\widetilde{f} : E_r \mapsto \mathbb{H}$ *such that* $\widetilde{f} = f$ *on the set* E.

Proof. We proceed as in the proof of Theorem 1 in [8], Section 10. Consider the series $p_0 + \sum_{k=0}^{\infty}(p_{k+1} - p_k)$. By Proposition 4.2, for the polynomial $p_{n+1} - p_n \in \mathbb{H}_{n+1}[q]$ we get the estimate

$$|p_{n+1}(q) - p_n(q)| \leq ||p_{n+1} - p_n||_E \; \Phi_E^{n+1}(q), \text{ for } q \in \mathbb{H}.$$

We have also

$$||p_{n+1} - p_n||_E \leq ||p_{n+1} - f||_E + ||p_n - f||_E$$

Chose $\varepsilon > 0$ such that

$$\|f - p_n\|_E \leq \left(\frac{1+\varepsilon}{R}\right)^n$$

for $n > N$, $N = N(\varepsilon)$ being sufficiently large. We get

$$|p_{n+1}(q) - p_n(q)| \leq 2\left(\frac{1+\varepsilon}{R}\right)^n \Phi_E^{n+1}(q), \text{ for } q \in \mathbb{H}$$

and

$$|p_{n+1}(q) - p_n(q)| \leq 2r\left(\frac{(1+\varepsilon)r}{R}\right)^n, \text{ for } q \in E_r.$$

Therefore the series $p_0 + \sum_{k=0}^{\infty}(p_{k+1} - p_k)$ converges uniformly on E_r. Since $p_0 + \sum_{k=0}^{n}(p_{k+1} - p_k) = p_n$ the sequence of polynomials $p_n \in \mathbb{H}_n[q]$ converges uniformly to the function f on the set E_r for $1 < r < R$. By Corrolary 3.6 function f is regular in the interior of the set E_R.

6. Polynomial approximation of regular functions in L^p spaces

Let μ be a finite Borel measure on a non-empty L-regular compact set $E \subset \mathbb{H}$. Proceeding as in [1] we say that the pair (E, μ) satisfies *Bernstein–Markov condition* (BM), if there exists p, $0 < p < \infty$ such that for any $\varepsilon > 0$ there exists $A = A(\varepsilon, p)$ such that

$$\|f\|_E \leq A(1+\varepsilon)^{\deg f}\|f\|_{\mu,p} \tag{9}$$

for all polynomials $f \in \mathbb{H}[q]$, where

$$\|f\|_{\mu,p} = \left(\int_E |f(q)|^p d\mu(q)\right)^{1/p}. \tag{10}$$

Using Hölder's inequality one may prove that if the pair (E, μ) satisfies (BM) for one exponent $p \in (0, \infty)$, then it satisfies (BM) for all exponents p, $0 < p < \infty$ (see [1], Remark 3.2). In particular, (10) defines a norm of $f : E \mapsto \mathbb{H}$ for $p \geq 1$.

Let $f : E \mapsto \mathbb{H}$ be a Borel function that has the bounded norm $\|f\|_{\mu,p} < \infty$. We propose the following version of Bernstein-Walsh-Siciak theorem in L^p spaces.

Theorem 6.1 *Let $E \subset \mathbb{H}$ be a L-regular non-empty compact set and let μ be a finite measure such, that Let (E, μ) satisfy (BM) for an exponent $p \geq 1$. and $f : E \mapsto \mathbb{H}$ be a Borel function with $\|f\|_{\mu,p} < \infty$. If $f_n \in \mathbb{H}_n[q]$ is a sequence of polynomials such that*

$$\limsup_{n\to\infty} ||f - f_n||_{\mu,p}^{1/n} \leq \frac{1}{R}$$

then f is regular μ almost everywhere in the interior of the set $E_R := \{q \in \mathbb{H} : \Phi_E(q) \leq R\}$, i.e. there exists $\tilde{f}$ regular in the interior of E_R and $\mu(E_R \cap \{q \in \mathbb{H} : \tilde{f}(q) \neq f(q)\}) = 0$.

Proof. The sequence $\|f_n\|_{\mu,p}$ is bounded because $\|f_n\|_{\mu,p}$ has the finite limit $\|f\|_{\mu,p} < \infty$. Fix $\varepsilon > 0$. By (BM) we have

$$||f_{n+1} - f_n||_E \leq A(1+\epsilon)^{n+1}||f_{n+1} - f_n||_{\mu,p}, \text{ for all } n = 1, 2, 3, \ldots,$$

$A = A(\varepsilon, \mu)$ being a constant depending on ε and μ only. This implies that the series

$$f_0(q) + \sum_{n=1}^{\infty}\big(f_n(q) - f_{n-1}(q)\big)$$

converges uniformly on compact subsets of the set $\{q : \Phi_E(q) < \frac{R}{1+\varepsilon}\}$ for arbitrary $\varepsilon > 0$. This gives the assertion of the theorem.

7. Regularity and leading terms of polynomials of best approximation

The relation between analyticity of a function $f : \mathbb{C}^n \supset E \mapsto \mathbb{C}$ and leading terms $\widehat{t}_n$ of polynomials of best approximation was studied in the case of one and several complex variables, see Theorem 2.1 in [2] and references given there. We will show that the regularity of a function $f : \mathbb{H} \supset E \mapsto \mathbb{H}$ inside a level curve $\{q \in \mathbb{H} : \Phi_E(q) < R\}$ of the polynomial extremal function Φ_E is related to norms of leading terms of the polynomials of best uniform polynomial approximation or best approximation in L^p norm.

We denote by $\widehat{p}_n(q) := q^n a_n$ the leading term of the polynomial $p_n(q) = a_0 + qa_1 + \cdots + q^n a_n \in \mathbb{H}_n[q]$.

Proposition 7.1 *Let $t_n(q) = \sum_{k=0}^{n} q^k a_k$ be the n-th polynomial of best uniform approximation of a function $f : E \mapsto \mathbb{H}$ bounded on a non-empty L-regular compact set $E \subset \mathbb{H}$, i.e.*

$$\|f - t_n\|_E = \inf\{\|f - p_n\|_E : p_n \in \mathbb{H}_n[q]\}.$$

If there exists $R > 1$ such that

$$\limsup_{n\to\infty} \|\widehat{t}_n\|_E^{1/n} \leq \frac{1}{R} \tag{11}$$

then function f is regular in the set $E_R := \{q \in \mathbb{H} : \Phi_E(q) < R\}$.

Proof. By the definition of polynomials of best uniform approximation we have

$$\|f - t_n\|_E \leq \|f - (t_{n+1} - \widehat{t}_{n+1})\|_E \leq \|f - t_{n+1}\|_E + \|\widehat{t}_{n+1}\|_E, \tag{12}$$

By (11) for $r \in (1, R)$ we obtain

$$\|\widehat{t}_{n+1}\|_E \leq \frac{1}{r^{n+1}}, \text{ for } n \geq N(r).$$

Repeating (12) we get

$$\begin{aligned}\|f - t_n\|_E &\leq \|\widehat{t}_{n+1}\|_E + \|f - t_{n+1}\|_E \\ &\leq \|\widehat{t}_{n+1}\|_E + \|\widehat{t}_{n+2}\|_E + \|f - t_{n+2}\|_E \leq \ldots \\ &\leq \frac{1}{r^{n+1}} + \frac{1}{r^{n+2}} + \cdots = \frac{1}{r-1}\frac{1}{r^n}.\end{aligned}$$

Hence $\limsup_{n\to\infty} \|f - t_n\|_E^{1/n} \leq \frac{1}{r}$ for any $r \in (1, R)$. By Theorem 5.1 we conclude that f is regular in E_R.

A similar proposition to the previous one we may obtain in the case of the polynomial approximation in the L^p norm if the pair (E, μ) satisfies condition (BM).

Proposition 7.2 *Let $E \subset \mathbb{H}$ be a non-empty regular compact set and μ be a finite Borel measure such that the pair (E, μ) satisfies (BM) for an exponent $p \geq 1$. Let $\tau_n(q) = \sum_{k=0}^{n} q^k a_k$ be the n-th polynomial of best approximation in L^p norm of a Borel function $f : E \mapsto \mathbb{H}$ with $\int_E |f(q)|^p d\mu(q) < \infty$, i.e.*

$$\|f - \tau_n\|_{\mu,p} = \inf \{ \|f - p_n\|_{\mu,p} : p_n \in \mathbb{H}_n[q] \}.$$

If there exists $R > 1$ such that

$$\limsup_{n\to\infty} \|\widehat{\tau}_n\|_{\mu,p}^{1/n} \leq \frac{1}{R} \tag{13}$$

then function f is regular μ almost everywhere in the set E_R.

Proof. Proceeding as in the proof of Proposition 7.1 we obtain

$$\|f - \tau_n\|_{\mu,p} \leq \|f - (\tau_{n+1} - \widehat{\tau}_{n+1})\|_{\mu,p} \leq \|f - \tau_{n+1}\|_{\mu,p} + \|\widehat{\tau}_{n+1}\|_{\mu,p}, \tag{14}$$

by the defintion of polynomials of best approximation in the L^p norm. By (13) for $r \in (1, R)$ we obtain

$$\|\widehat{\tau}_{n+1}\|_{E,\mu} \leq \frac{1}{r^{n+1}}, \text{ for } n \geq N(r).$$

Repeating (14) we get

$$\begin{aligned}\|f - \tau_n\|_{E,\mu} &\leq \|\widehat{\tau}_{n+1}\|_{E,\mu} + \|f - \tau_{n+1}\|_{E,\mu} \\ &\leq \|\widehat{\tau}_{n+1}\|_{E,\mu} + \|\widehat{\tau}_{n+2}\|_{E,\mu} + \|f - \tau_{n+2}\|_{E,\mu} \leq \dots \\ &\leq \frac{1}{r^{n+1}} + \frac{1}{r^{n+2}} + \cdots = \frac{1}{r-1}\frac{1}{r^n}.\end{aligned}$$

Hence $\limsup_{n\to\infty} \|f - \tau_n\|_{E,\mu}^{1/n} \leq \frac{1}{r}$ for any $r \in (1, R)$. By Theorem 6.1 we conclude that f is regular μ almost everywhere in E_R.

8. Factors of polynomials approximating regular functions

We start with an observation that the points q_{nk}, see (15), in the factorisation of partial sums of power series of a regular function in the ball $B(0, R) := \{q \in \mathbb{H} : |q| < R\}$ have to lay outside this ball.

Proposition 8.1 *Let $s_n(q) = \sum_{k=0}^{n} q^k a_k$ be the n-th partial sum of the power series $\sum_{k=0}^{\infty} q^k a_k$ and let $q_{nk} \in \mathbb{H}$, $k = 1, 2, \ldots, n$ be a sequence of quaternions in the factorisation of s_n*

$$s_n(q) = (q - q_{n1}) * (q - q_{n2}) * \cdots * (q - q_{nn}) a_n. \tag{15}$$

If the points lay outside the ball $B(0, R)$ for $R > 0$ and for $n \geq n_0$ then s_n converges to a function f regular in $B(0, R)$.

Proof. Observe that

$$\begin{aligned} |a_0| = |s_n(0)| &= |(0 - q_{n1}) * (0 - q_{n2}) * \cdots * (0 - q_{nn}) * a_n| \\ &= |q_{n1}\, q_{n2} \ldots q_{nn}\, a_n| = |q_{n1}|\, |q_{n2}| \ldots |q_{nn}|\, |a_n| \geq R^n |a_n| \end{aligned} \tag{16}$$

Hence $\limsup_{n\to\infty} |a_n|^{1/n} \leq \frac{1}{R}$. This implies the uniform converence of the power series $s_n(q) = \sum_{k=0}^{n} q^k a_k$ to a function f regular in $B(0, R)$, see [5], Theorem 1.6.

The classical Jentzsch's theorem on the complex plane states that if $s_n(z) := \sum_{k=0}^{n} a_k z^k$ are partial sums of the series $f(z) = \sum_{k=0}^{\infty} a_k z^k$ with radius of convergence 1, then each point on the circle of convergence $|z| = 1$ is a limit point of zeros of the polynomials s_n. Hence the natural question arises.

Question 8.2 Is every point of the set $|q| = R$ an accumulation point of quaternions q_{nk}, where $s_n(q) = (q - q_{n1}) * (q - q_{n2}) * \cdots * (q - q_{nn})a_n$, if the function $f(q) = \sum_{k=0}^{\infty} q^k a_k$ is regular in the set $|q| < R$ and is not regular in the larger ball $|q| < R_1$, for $R_1 > R$?

By [5], Theorem 2.11, the series

$$f(q) = \sum_{k=0}^{\infty} (q - c)^{*n} a_n$$

converges on compact subsets of σ-ball $\Sigma(c, R)$, where $\frac{1}{R} := \limsup_{n\to\infty} |a_n|^{1/n}$, and it does not converge at any point of $\mathbb{H} \setminus \overline{\Sigma(c, R)}$. Hence the next question arises.

Question 8.3 Is every point of the set $\{q \in \mathbb{H} : \sigma(c, q) = R\}$ an accumulation point of quaternions q_{nk}, where

$$s_n(q) = \sum_{k=0}^{n} (q - c)^{*k} a_k = (q - q_{n1}) * (q - q_{n2}) * \cdots * (q - q_{nn})a_n?$$

Partial sums of the power series $\sum_{k=0}^{\infty} q^k a_k$ may be replaced by polynomials of best uniform approximation or best approximation in L^p norm.

Proposition 8.4 *Let $t_n(q) = \sum_{k=0}^{n} q^k a_{nk} \in \mathbb{H}_n[q]$ be the sequence of polynomials of best uniform approximation of a function $f : \overline{B(0, r)} \mapsto \mathbb{H}$ bounded on the closed ball $\overline{B(0, r)} = \{q \in \mathbb{H} : |q| \leq r\}$, for $r > 0$. If the quaternions q_{nk} in the factors*

$$t_n(q) = (q - q_{n1}) * (q - q_{n2}) * \cdots * (q - q_{nn})a_{nn}. \tag{17}$$

lay outside the ball $B(0, R)$, for $R > r$, then function f is regular in the ball $B(0, R)$.

Proof. Observe that $\|\widehat{t_n}\|_{\overline{B(0,r)}} = \|q^n a_{nn}\|_{\overline{B(0,r)}} = |a_{nn}|r^n$. Function f is bounded, so the sequence $|t_n(0)|$ is bounded as well. Moreover

$$|t_n(0)| = |(0 - q_{n1}) * (0 - q_{n2}) * \cdots * (0 - q_{nn})a_{nn}| \geq R^n |a_{nn}|.$$

Hence $\|\widehat{t_n}\|^{1/n}_{\overline{B(0,r)}} = |a_{nn}|^{1/n} r \leq \frac{r}{R}|t_n(0)|^{1/n}$ and $\limsup_{n\to\infty} \|\widehat{t_n}\|^{1/n}_{\overline{B(0,r)}} \leq \frac{r}{R}$. By Proposition 7.1 function f is regular in the set

$$\left\{ q \in \mathbb{H} : \Phi_{\overline{B(0,r)}}(q) < \frac{R}{r} \right\} = B(0, R),$$

as $\Phi_{\overline{B(0,r)}}(q) = \max\{1, \frac{|q|}{r}\}$, see Proposition 4.1.

Proposition 8.5 *Let $E := \overline{B(0,r)}$, $r > 0$, and let μ be a finite Borel measure on E such that (E, μ) satisfies (BM) for an exponent $p \geq 1$. Let $\tau_n(q) = \sum_{k=0}^{n} q^k a_{nk} \in \mathbb{H}_n[q]$ be the sequence of polynomials of best approximation in L^p norm of function $f : E \mapsto \mathbb{H}$, $\int_E |f(q)|^p d\mu(q) < \infty$, i.e.*

$$\|f - \tau_n\|_{E,\mu} = \inf\{\|f - p_n\|_{E,\mu} : p_n \in \mathbb{H}_n[q]\}.$$

If the quaternions q_{nk} in the factors

$$\tau_n(q) = (q - q_{n1}) * (q - q_{n2}) * \cdots * (q - q_{nn})a_{nn}. \tag{18}$$

lay outside the ball $B(0, R)$, for $R > r$, then function f is regular μ almost everywhere in the ball $B(0, R)$.

Proof. Proceeding as in the proof of Proposition 8.4 we have

$$\|\widehat{\tau_n}\|_{\overline{B(0,r)}} = \|q^n a_{nn}\|_{\overline{B(0,r)}} = |a_{nn}|r^n.$$

Fix $\epsilon > 0$ and note that by (BM) we have

$$\|p_n\|_{\overline{B(0,r)}} \leq A(1 + \epsilon)^n \|p_n\|_{\mu,p}, \text{ for any polynomial } p_n \in \mathbb{H}_n[q].$$

Thus

$$|\tau_n(0)| \leq \|\tau_n\|_{\overline{B(0,r)}} \leq A(1 + \epsilon)^n \|\tau_n\|_{\mu,p} \tag{19}$$

$A = A(\epsilon, p)$ being a constant. The sequence $\|\tau_n\|_{\mu,p}$ is bounded because it has the finite limit $\|f\|_{\mu,p} < \infty$. Moreover

$$|\tau_n(0)| = |(0 - q_{n1}) * (0 - q_{n2}) * \cdots * (0 - q_{nn})a_{nn}| \geq R^n |a_{nn}|. \tag{20}$$

By (19) and (20) we have

$$\|\widehat{\tau_n}\|^{1/n}_{\overline{B(0,r)}} = |a_{nn}|^{1/n} r \leq (1 + \epsilon)\frac{r}{R}\left(A\|\tau_n\|_{\mu,p}\right)^{1/n}.$$

Thus

$$\limsup_{n\to\infty} \|\widehat{t_n}\|^{1/n}_{B(0,r)} \le (1+\epsilon)\frac{r}{R}, \text{ for any } \epsilon > 0.$$

By Proposition 7.1 function f is regular μ almost everywhere in the set

$$\left\{q \in \mathbb{H} : \Phi_{\overline{B(0,r)}}(q) < \frac{R}{r}\right\} = B(0,R),$$

as $\Phi_{\overline{B(0,r)}}(q) = \max\{1, \frac{|q|}{r}\}$, see Proposition 4.1.

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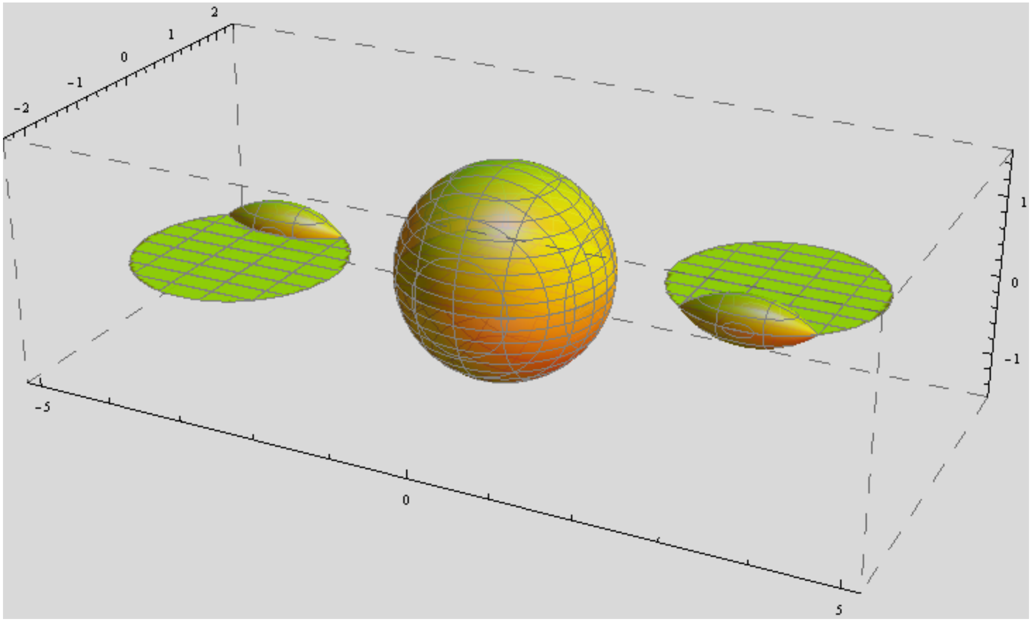

Figure 2: The sets $\Sigma(c_i, R_i) \cap \{x_0 + ix_1 + jx_2 + kx_3 \in \mathbb{H} : x_3 = 0\}$ with $c_1 = (-3, -1, 0, 0)$, $c_2 = (0, 0, 0, 0)$, $c_3 = (3, 1, 0, 0)$ and $R_1 = R_2 = R_3 = \frac{13}{10}$. Note that $c_2 = 0$ belongs to every complex line L_I, thus $\sigma(c_2, q) = |c_2 - q|$ and $\Sigma(c_2, R_2) = B(c_2, R_2)$.

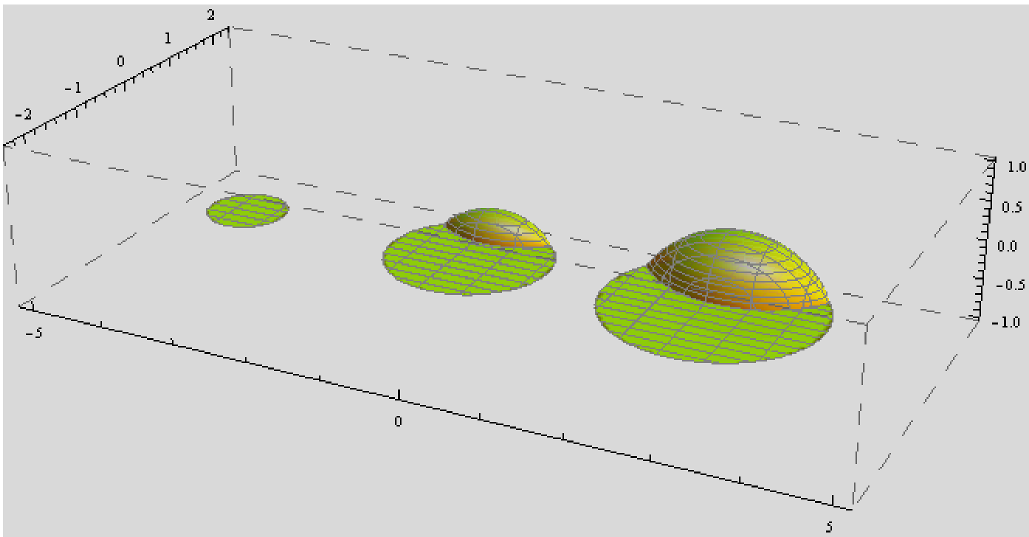

Figure 3: The sets $\Sigma(c_i, R_i) \cap \{x_0 + ix_1 + jx_2 + kx_3 \in \mathbb{H} : x_3 = 0\}$ with $c_1 = (-3, -\frac{7}{10}, 0, 0)$, $c_2 = (0, -\frac{7}{10}, 0, 0)$, $c_3 = (3, -\frac{7}{10}, 0, 0)$ and $R_1 = \frac{1}{2}$, $R_2 = 1$, $R_3 = \frac{13}{10}$.