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Positive solutions for nonlinear Robin problems with convection

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Original article

Abstract

We consider a Robin problem driven by the p-Laplacian and with a reaction which is gradient dependent (convection). Using truncations and perturbations, we show that the problem has at least one positive smooth solution.

Introduction

Let $\Omega \subseteq \mathbb{R}^N$ be a bounded domain with C^2-boundary $\partial\Omega$. In this paper we study the following nonlinear Robin problem with gradient dependence (convection):

$$\left\{ \begin{array}{ll} -\Delta_p u(z) = f(z, u(z), Du(z)) & \text{in } \Omega, \\ \frac{\partial u}{\partial n_p} + \beta(z) u^{p-1} = 0, & \text{in } \partial\Omega, \ u \geqslant 0, \ 1 < p < +\infty. \end{array} \right. \tag{1.1}$$

Here Δ_p denotes the p-Laplace differential operator defined by

$$\Delta_p u = \operatorname{div}\left(|Du|^{p-2} Du\right) \quad \forall u \in W^{1,p}(\Omega).$$

The reaction $f(z, x, y)$ is a Carathéodory function. No global growth restriction is imposed on $f(z, \cdot, \cdot)$. All the restrictions are local and concern the structure of $f(z, \cdot, \cdot)$ near zero. A similar problem was investigated recently by Bai-Gasiński-Papageorgiou [2]. There the differential operator is more general and

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- convection
- pseudomonotone operator
- strong comparison principle
- nonlinear regularity

Authors contributions

A – Conceptualization
B – Methodology
C – Formal analysis
D – Software
E – Investigation
F – Data duration
G – Visualization
H – Writing – original draft preperation
I – Writing, reviewing & editing
J – Project administration
K – Funding acquisition

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it is in general nonhomogeneous. However, the conditions on the reaction are more restrictive and global. In addition, the approach there is different and it uses the so called "freezing method". Namely we freeze the gradient variable. Then the resulting problem is variational and can be solved using the minimax methods (critical point theory). We show that the "frozen problem" has a smallest positive solution $\widehat{u}_v$ depending on the frozen variable v. We consider the map $v \to \widehat{u}_v$ and use fixed point theory to produce a solution of the original problem. In contrast here we use truncation techniques and the theory of nonlinear operators of monotone type. Nonlinear operator theory was also used by Gasiński-Krech-Papageorgiou [4] but for a problem with a reaction which is globally restricted. Finally we mention also the works of Bai [1], Faraci-Motreanu-Puglisi [3], Gasiński-Papageorgiou [6], Papageorgiou-Rădulescu-Repovš [11], Zeng-Papageorgiou [12] (singular problems with convection), all using the frozen variable method.

We mention that in the boundary condition $\frac{\partial u}{\partial n_p}$ denotes the conormal derivative of u corresponding to the p-Laplacian. This condition is interpreted using the nonlinear Green's identity (see, for example, Gasiński-Papageorgiou [5, p. 221]). When $u \in C^1(\overline{\Omega})$, then

$$\frac{\partial u}{\partial n_p} = |Du|^{p-2}(Du, n)_{\mathbb{R}^N} = |Du|^{p-2}\frac{\partial u}{\partial n},$$

with n being the outward unit normal on $\partial\Omega$.

Mathematical background hypotheses

The following spaces are important in the analysis of problem (1.1):

$$W^{1,p}(\Omega), \quad C^1(\overline{\Omega}) \quad \text{and} \quad L^p(\partial\Omega).$$

By $\|\cdot\|$ we denote the norm of the Sobolev space $W^{1,p}(\Omega)$ defined by

$$\|u\| = \left(\|u\|_p^p + \|Du\|_p^p\right)^{\frac{1}{p}} \quad \forall u \in W^{1,p}(\Omega).$$

The Banach space $C^1(\overline{\Omega})$ is ordered with positive (order) cone

$$C_+ = \{u \in C^1(\overline{\Omega}) : \ u(z) \geqslant 0 \text{ for all } z \in \overline{\Omega}\}.$$

This cone has a nonempty interior given by

$$\text{int}C_+ = \{u \in C_+ : \ u(z) > 0 \text{ for all } z \in \overline{\Omega}\}.$$

We will also use another cone in $C^1(\overline{\Omega})$ defined by

$$D_+ = \left\{u \in C_+ : \ u(z) > 0 \text{ for all } z \in \Omega, \ \frac{\partial u}{\partial n}\bigg|_{\partial\Omega \cap u^{-1}(0)} < 0\right\}.$$

On $\partial\Omega$ we consider the $(N-1)$-dimensional Hausdorff (surface) measure σ. With this measure on $\partial\Omega$, we can define in the usual way the boundary Lebesgue space $L^\tau(\partial\Omega)$ $(1 \leqslant \tau \leqslant \infty)$. We know that there exists a unique continuous linear map $\gamma_0\colon W^{1,p}(\Omega) \longrightarrow L^p(\partial\Omega)$ known as the "trace map" such that $\gamma_0(u) = u|_{\partial\Omega}$ for all $u \in W^{1,p}(\Omega) \cap C(\overline{\Omega})$. Hence the trace map extends the notion of "boundary values" to all Sobolev functions. This map is compact into $L^r(\partial\Omega)$ for all $r \in [1, \frac{(p-1)N}{N-p})$, if $p < N$ and into $L^r(\partial\Omega)$ for all $1 \leqslant r < \infty$ if $N \leqslant p$. We have

$$\ker\gamma_0 = W_0^{1,p}(\Omega) \quad \text{and} \quad \operatorname{im}\gamma_0 = W^{\frac{1}{p'},p}(\partial\Omega)$$

$(\frac{1}{p} + \frac{1}{p'} = 1)$. For the sake of notational simplicity, in the sequel we drop the use of the trace map γ_0. All restrictions of Sobolev functions on $\partial\Omega$ are understood in the sense of traces.

Consider the following nonlinear eigenvalue problem:

$$\begin{cases} -\Delta_p u(z) = \widehat{\lambda}|u(z)|^{p-2}u(z) & \text{in } \Omega, \\ \frac{\partial u}{\partial n_p} + \beta(z)u^{p-2}u = 0, & \text{in } \partial\Omega. \end{cases} \tag{2.1}$$

We consider the following conditions on the boundary coefficient β:

$\underline{H_0}$: $\beta \in C^{0,\alpha}(\partial\Omega)$ for some $\alpha \in (0,1)$ and $\beta(z) \geqslant 0$ for all $z \in \partial\Omega$.

Remark 2.1. If $\beta \equiv 0$, then we have the Neumann problem.

We say that $\widehat{\lambda}$ is an eigenvalue of (2.1), if the problem admits a nontrivial solution $\widehat{u} \in W^{1,p}(\Omega)$ known as an eigenfunction corresponding to the eigenvalue $\widehat{\lambda}$. There is a smallest eigenvalue $\widehat{\lambda}_1$ with the following properties:

- $\widehat{\lambda}_1 \geqslant 0$ and $\widehat{\lambda}_1 = 0$ if $\beta \equiv 0$ (Neumann problem), $\widehat{\lambda}_1 > 0$ if $\beta \not\equiv 0$;
- $\widehat{\lambda}_1$ is isolated in the spectrum $\widehat{\sigma}(p)$ of (2.1), that is, there exists $\varepsilon > 0$ such that $(\widehat{\lambda}_1, \widehat{\lambda}_1 + \varepsilon) \cap \widehat{\sigma}(p) = \emptyset$;
- $\widehat{\lambda}_1$ is simple (that is, if $\widehat{u}, \widehat{v}$ are eigenfunctions corresponding to $\widehat{\lambda}_1$, then $\widehat{u} = \mu\widehat{v}$ for some $\mu \in \mathbb{R} \setminus \{0\}$);
- we have

$$\widehat{\lambda}_1 = \inf_{u \in W^{1,p}(\Omega)\setminus\{0\}} \frac{\gamma(u)}{\|u\|_p^p}, \tag{2.2}$$

where

$$\gamma(u) = \|Du\|_p^p + \int_{\partial\Omega} \beta(z)|u|^p \, d\sigma \quad \forall u \in W^{1,p}(\Omega).$$

In (2.2) the infimum is realized on the corresponding one dimensional eigenspace, the elements of which have constant sign. The nonlinear regularity theory of Lieberman [7] implies that every eigenfunction of (2.1) belongs in $C^1(\overline{\Omega})$. By $\widehat{u}_1$ we denote the L^p-normalized (that is, $\|\widehat{u}_1\|_p = 1$) positive eigenfunction corresponding to $\widehat{\lambda}_1$. Then the nonlinear regularity theory and the nonlinear maximum principle (see Gasiński-Papageorgiou [5, p. 738]), imply that $\widehat{u}_1 \in \operatorname{int}C_+$. For details we refer to Papageorgiou-Rădulescu [9].

Let $A\colon W^{1,p}(\Omega) \longrightarrow W^{1,p}(\Omega)^*$ be the nonlinear operator defined by

$$\langle A(u), h\rangle = \int_\Omega |Du|^{p-2}(Du, Dh)_{\mathbb{R}^N}\, dz \quad \forall u, h \in W^{1,p}(\Omega).$$

This operator has the following properties (see Gasiński-Papageorgiou [5]).

Proposition 2.2. *The operator A is bounded (that is, maps bounded sets to bounded sets), continuous, monotone (hence maximal monotone too) and of type $(S)_+$, that is, "$u_n \xrightarrow{w} u$ in $W^{1,p}(\Omega)$ and $\limsup_{n\to+\infty}\langle A(u_n), u_n - u\rangle \leqslant 0$ imply that $u_n \longrightarrow u$ in $W^{1,p}(\Omega)$ as $n \to \infty$".*

Let X be a reflexive Banach space, X^* its topological dual and by $\langle\cdot,\cdot\rangle_X$ we denote the duality brackets for the pair (X, X^*). We say that a nonlinear operator $\widehat{V}\colon X \longrightarrow X^*$ is "pseudomonotone", if it has the following property: "If $u_n \xrightarrow{w} u$ in X, $A(u_n) \xrightarrow{w} y^*$ in X^* and

$$\limsup_{n\to+\infty}\langle A(u_n), u_n - u\rangle_X \leqslant 0,$$

then $y^* = A(u)$ and $\langle A(u_n), u_n\rangle_X \longrightarrow \langle A(u), u\rangle_X$" (see Gasiński-Papageorgiou [5, p. 330]).

Now, we introduce the hypotheses on the reaction term $f(z, x, y)$:

$\underline{H_1}$: $f\colon \Omega \times \mathbb{R} \times \mathbb{R}^N \longrightarrow \mathbb{R}$ is a Carathéodory function such that $f(z, 0, 0) = 0$ for almost all $z \in \Omega$ and

(i) there exists $c_+ > 0$ such that

$$f(z, c_+, 0) \leqslant 0 \quad \text{for a.a. } z \in \Omega;$$

(ii) there exist a function $\eta \in L^\infty(\Omega)$ and $\delta_0 \in (0, c_+)$ such that

$$\eta(z) \geqslant \widehat{\lambda}_1 \quad \text{for a.a. } z \in \Omega, \quad \eta \not\equiv \widehat{\lambda}_1$$

and for every $\varepsilon > 0$, there exists $c_\varepsilon > 0$ such that

$$f(z, x, y) \geqslant (\eta(z) - \varepsilon)x^{p-1} - c_\varepsilon x^{r-1}$$

for almost all $z \in \Omega$, all $0 \leqslant x \leqslant \delta_0$, all $y \in \mathbb{R}^N$, with $r \in (p, p^*)$ (recall that $p^* = \frac{Np}{N-p}$ if $p < N$ and $p = +\infty$ if $N \leqslant p$);

(iii) for every $\varrho > 0$, there exists $a_\varrho \in L^\infty(\Omega)$ such that

$$|f(z, x, y)| \leqslant a_\varrho(z)$$

for almost all $z \in \Omega$, all $0 \leqslant x \leqslant c_+$, all $|y| \leqslant \varrho$.

Remark 2.3. Hypotheses $H_1(ii)$ is satisfied if the following is true

$$\liminf_{x\to 0^+} \frac{f(z,x,y)}{x^{p-2}} \geqslant \eta(z)$$

uniformly for almost all $z \in \Omega$, all $y \in \mathbb{R}^N$. Note that all conditions in H_1 concern the behaviour of $f(z,\cdot,y)$ near zero.

The following function satisfies hypotheses H_1:

$$f(z,x,y) = f_0(x) + \xi(z)|y|^{\vartheta},$$

with $\xi \in L^\infty(\Omega)$, $\xi(z) \geqslant 0$ for almost all $z \in \Omega$, $\vartheta \geqslant 1$ and $f_0 \in C(\mathbb{R}_+;\mathbb{R})$ satisfies

$$\liminf_{x\to 0^+} \frac{f_0(x)}{x^{p-1}} \geqslant \eta > \widehat{\lambda}_1$$

and there exists $c_+ > 0$ such that $f_0(c_+) \leqslant 0$.

An auxiliary problem

In this section we examine the following auxiliary nonlinear Robin problem

$$\begin{cases} -\Delta_p u(z) = (\eta(z)-\varepsilon)u(z)^{p-1} - c_\varepsilon u(z)^{r-1} & \text{in } \Omega, \\ \frac{\partial u}{\partial n_p} + \beta(z)u^{p-1} = 0, & \text{in } \partial\Omega,\ u \geqslant 0,\ 1 < p < +\infty. \end{cases} \tag{3.1}$$

Proposition 3.1. *If hypotheses H_0 hold, then for all $\varepsilon > 0$ small problem* (3.1) *admits a unique solution $\widetilde{u}_0 \in \operatorname{int} C_+$.*

Proof. Let $\psi\colon W^{1,p}(\Omega) \longrightarrow \mathbb{R}$ be the C^1-functional defined by

$$\psi(u) = \frac{1}{p}\gamma(u) + \frac{1}{p}\|u^-\|_p^p + \frac{c_\varepsilon}{r}\|u^+\|_r^r - \frac{1}{p}\int_\Omega (\eta(z)-\varepsilon)(u^+)^p\,dz.$$

Since $p < r$, we see that ψ is coercive. Also using the Sobolev embedding theorem and the compactness of the trace map, we show that ψ is sequentially weakly lower semicontinuous. So, invoking the Weierstrass-Tonelli theorem, we can find $\widetilde{u}_0 \in W^{1,p}(\Omega)$ such that

$$\psi(\widetilde{u}_0) = \inf_{u\in W^{1,p}(\Omega)} \psi(u). \tag{3.2}$$

Let $t > 0$ and recall that $\widehat{u}_1 \in \operatorname{int} C_+$ (see Section 2). We have

$$\psi(t\widehat{u}_1) = \frac{t^p}{p}\left(\int_\Omega (\widehat{\lambda}_1 - \eta(z))\widehat{u}_1^p\,dz + \varepsilon\right) + \frac{t^r c_\varepsilon}{r}\|\widehat{u}_1\|_r^r \tag{3.3}$$

(recall that $\gamma(\widehat{u}_1) = \widehat{\lambda}_1\|\widehat{u}_1\|_p^p$; see (2.2) and $\|\widehat{u}_1\|_p = 1$). Hypothesis $H_1(ii)$ implies that

$$\widehat{c} = \int_\Omega (\eta(z) - \widehat{\lambda}_1)\widehat{u}_1^p\,dz > 0$$

(since $\widehat{u}_1 \in \text{int}C_+$). Choosing $\varepsilon \in (0, \widehat{c})$, from (3.3), we have

$$\psi(t\widehat{u}_1) \leqslant c_1 t^r - c_2 t^p,$$

for some $c_1, c_2 > 0$. Since $r > p$, for $t \in (0,1)$ small we have

$$\psi(t\widehat{u}_1) < 0,$$

so

$$\psi(\widetilde{u}_0) < 0 = \psi(0)$$

(see (3.2)), thus $\widetilde{u}_0 \neq 0$. From (3.2) we have

$$\psi'(\widetilde{u}_0) = 0,$$

so

$$\begin{aligned} \langle A(\widetilde{u}_0), h\rangle + \int_{\partial\Omega} \beta(z)|\widetilde{u}_0|^{p-2}\widetilde{u}_0 h\, dz - \int_\Omega (\widetilde{u}_0^-)^{p-1} h\, dz \\ = \int_\Omega (\eta(z) - \varepsilon)(\widetilde{u}_0^+)^{p-1} h\, dz - c_\varepsilon \int_\Omega (\widetilde{u}_0^+)^{r-1} h\, dz \quad \forall h \in W^{1,p}(\Omega). \quad (3.4) \end{aligned}$$

In (3.4) above we choose $h = -\widetilde{u}_0^- \in W^{1,p}(\Omega)$. Then

$$\gamma(\widetilde{u}_0^-) + \|\widetilde{u}_0^-\|_p^p = 0,$$

so $\widetilde{u}_0 \geqslant 0$ and $\widetilde{u}_0 \neq 0$. Then from (3.4) and Green's identity, we have

$$\begin{cases} -\Delta_p \widetilde{u}_0(z) = (\eta(z) - \varepsilon)\widetilde{u}_0(z)^{p-1} - c_\varepsilon \widetilde{u}_0(z)^{r-1} & \text{in } \Omega, \\ \frac{\partial \widetilde{u}_0}{\partial n_p} + \beta(z)\widetilde{u}_0^{p-1} = 0, & \text{in } \partial\Omega \end{cases} \quad (3.5)$$

(see Papageorgiou-Rădulescu [9]). From (3.5) and Papageorgiou-Rădulescu [8, Proposition 2.10], we have

$$\widetilde{u}_0 \in L^\infty(\Omega).$$

Then applying Theorem 1 of Lieberman [7], we obtain

$$\widetilde{u}_0 \in C_+ \setminus \{0\}.$$

From (3.5), we have

$$\Delta_p \widetilde{u}_0(z) \leqslant \left(c_\varepsilon \|\widetilde{u}_0\|_\infty^{r-p}\right)\widetilde{u}_0(z)^{p-1} \quad \text{for a.a. } z \in \Omega.$$

Then the nonlinear strong maximum principle (see Gasiński-Papageorgiou [5, p. 738]), implies that

$$\widetilde{u}_0 \in \text{int}C_+.$$

Next we show that this positive solution of (3.1) is unique. So, let $\widetilde{v}_0 \in W_0^{1,p}(\Omega)$ be another positive solution of (3.1). Again we have $\widetilde{v}_0 \in \text{int}C_+$. Let $t > 0$ be the biggest positive real such that

$$t\widetilde{v}_0 \leqslant \widetilde{u}_0. \quad (3.6)$$

Let $\varrho = \|\widetilde{u}_0\|_\infty$ and let $\widehat{\xi}_\varrho > 0$ be such that for almost all $z \in \Omega$, the function

$$x \longmapsto (\eta(z) - \varepsilon)x^{p-1} - c_\varepsilon x^{r-1} + \widehat{\xi}_\varrho x^{p-1}$$

is nondecreasing on $[0, \varrho]$. Suppose that $t \in (0,1)$. Using (3.6) and recalling the choice of $\widehat{\xi}_\varrho$ and that $r > p$, $t \in (0,1)$, we have

$$\begin{aligned} & -\Delta_p(t\widetilde{v}_0) + \widehat{\xi}_p(t\widetilde{v}_0)^{p-1} \\ = \; & t^{p-1}\big(-\Delta_p\widetilde{v}_0 + \widehat{\xi}_\varrho\widetilde{v}_0^{p-1}\big) \\ = \; & t^{p-1}\big((\eta(z)-\varepsilon)\widetilde{v}_0^{p-1} - c_\varepsilon\widetilde{v}_0^{r-1} + \widehat{\xi}_\varrho\widetilde{v}_0^{p-1}\big) \\ < \; & (\eta(z)-\varepsilon)(t\widetilde{v}_0)^{p-1} - c_\varepsilon(t\widetilde{v}_0)^{r-1} + \widehat{\xi}_\varrho(t\widetilde{v}_0)^{p-1} \\ \leqslant \; & (\eta(z)-\varepsilon)\widetilde{u}_0^{p-1} - c_\varepsilon\widetilde{u}_0^{r-1} + \widehat{\xi}_\varrho\widetilde{u}_0^{p-1} \\ = \; & -\Delta_p\widetilde{u}_0 + \widehat{\xi}_\varrho\widetilde{u}_0^{p-1}. \end{aligned} \tag{3.7}$$

Since $\widetilde{u}_0 \in \text{int}C_+$, we have that $0 < \widetilde{m}_0 = \min\limits_{\overline{\Omega}} \widetilde{v}_0$. Hence

$$c_\varepsilon(t^{p-1} - t^{r-1})\widetilde{v}_0^{r-1} \geqslant c_\varepsilon(t^{p-1} - t^{r-1})\widetilde{m}_0^{r-1} > 0.$$

Then from (3.7) and Proposition 2.10 of Papageorgiou-Rădulescu-Repovš [10] (the strong comparison principle), we have

$$\widetilde{u}_0 - t\widetilde{v}_0 \in D_+,$$

which contradicts the maximality of $t > 0$. This means that $t \geqslant 1$ and so

$$\widetilde{v}_0 \leqslant \widetilde{u}_0$$

(see (3.6)). If in the above argument, we interchange the roles of $\widetilde{u}_0$ and $\widetilde{v}_0$, we obtain

$$\widetilde{u}_0 \leqslant \widetilde{v}_0$$

and so

$$\widetilde{u}_0 = \widetilde{v}_0.$$

This proves the uniqueness of the positive solution $\widetilde{u}_0 \in \text{int}C_+$. □

We choose $t \in (0,1)$ small so that

$$t\widetilde{u}_0(z) \in (0, \delta_0] \quad \forall z \in \overline{\Omega}. \tag{3.8}$$

Let $\widetilde{u} = t\widetilde{u}_0 \in \text{int}C_+$. Using also Proposition 3.1 and the fact that $t \in (0,1)$, we have

$$\begin{aligned} -\Delta_p\widetilde{u}(z) &= t^{p-1}(-\Delta_p\widetilde{u}_0(z)) \\ &= t^{p-1}\big((\eta(z)-\varepsilon)\widetilde{u}_0(z)^{p-1} - c_\varepsilon\widetilde{u}_0(z)^{r-1}\big) \\ &\leqslant (\eta(z)-\varepsilon)(tu_0(z))^{p-1} - c_\varepsilon(tu_0(z))^{r-1} \\ &\leqslant f(z, \widetilde{u}(z), D\widetilde{u}(z)) \end{aligned} \tag{3.9}$$

(see (3.8) and hypothesis $H_1(ii)$). Also with $\widehat{\xi}_\varrho > 0$ as in the proof of Proposition 2, using (3.8), recalling that $\delta_0 < c_+$ and using hypothesis $H_1(ii)$, we have

$$\begin{aligned} & -\Delta_p \widetilde{u}(z) + \widehat{\xi}_\varrho \widetilde{u}(z)^{p-1} \\ \leqslant\ & (\eta(z) - \varepsilon)\widetilde{u}(z)^{p-1} - c_\varepsilon \widetilde{u}(z)^{r-1} + \widehat{\xi}_\varrho \widetilde{u}(z)^{p-1} \\ \leqslant\ & (\eta(z) - \varepsilon) c_+^{p-1} - c_\varepsilon c_+^{r-1} + \widehat{\xi}_\varrho c_+^{p-1} \\ \leqslant\ & f(z, c_+, 0) + \widehat{\xi}_\varrho c_+^{p-1} \\ \leqslant\ & -\Delta_p c_+ + \widehat{\xi}_\varrho c_+^{p-1}, \end{aligned}$$

so

$$\widetilde{u}(z) < c_+ \quad \forall z \in \overline{\Omega} \tag{3.10}$$

(see Papageorgiou-Rădulescu-Repovš [10, Proposition 2.10]).

Positive solutions

In this section using the theory of nonlinear operators of monotone type, we show the existence of a positive smooth solution for problem (1.1).

Theorem 4.1. *If hypotheses H_0 and H_1 hold, then problem* (1.1) *has a positive solution $\widehat{u} \in \operatorname{int} C_+$ and*

$$\widehat{u}(z) < c_+ \quad \forall z \in \overline{\Omega}.$$

Proof. Recall that $\widetilde{u} \leqslant c_+$ (see (3.10)).

Next consider the continuous map $\tau\colon W^{1,p}(\Omega) \longrightarrow W^{1,p}(\Omega)$ defined by

$$\tau(u)(z) = \begin{cases} \widetilde{u}(z) & \text{if} \quad u(z) < \widetilde{u}(z), \\ u(z) & \text{if} \quad \widetilde{u}(z) \leqslant u(z) \leqslant c_+, \\ c_+ & \text{if} \quad c_+ < u(z). \end{cases} \tag{4.1}$$

Also let $\widehat{f}\colon \Omega \times \mathbb{R} \times \mathbb{R}^N \longrightarrow \mathbb{R}$ be the following perturbation of f:

$$\widehat{f}(z, x, y) = f(z, x, y) + |x|^{p-2}x.$$

This is a Carathéodory function.

We consider the nonlinear operator $V\colon W^{1,p}(\Omega) \longrightarrow W^{1,p}(\Omega)^*$ defined by

$$\begin{aligned} \langle V(u), h\rangle \ =\ & \langle A(u), h\rangle + \int_\Omega |u|^{p-2}uh\,dz + \int_{\partial\Omega} \beta(z)|u|^{p-2}uh\,d\sigma \\ & - \int_\Omega f(z, \tau(z), D\tau(z))h\,dz \quad \forall h \in W^{1,p}(\Omega). \end{aligned}$$

Clearly V is bounded, continuous (see Proposition 2.2). Also assume that

$$\begin{cases} u_n \xrightarrow{w} u \quad \text{in } W^{1,p}(\Omega), \quad V(u_n) \xrightarrow{w} u^* \quad \text{in } W^{1,p}(\Omega)^*, \\ \limsup\limits_{n \to +\infty} \langle V(u_n), u_n - u\rangle \leqslant 0. \end{cases} \tag{4.2}$$

From (4.2) we have

$$u_n \longrightarrow u \quad \text{in } L^p(\Omega) \quad \text{and in } L^p(\partial\Omega).$$

Therefore we have

$$\int_\Omega |u_n|^{p-2}u_n(u_n-u)\,dz \longrightarrow 0, \quad \int_{\partial\Omega} \beta(z)|u_n|^{p-2}u_n(u_n-u)\,d\sigma \longrightarrow 0 \quad (4.3)$$

and

$$\int_\Omega f(z,\tau(u_n),D\tau(u_n))(u_n-u)\,dz \longrightarrow 0 \quad (4.4)$$

(see (4.1) and hypothesis $H_1(iii)$). From (4.2), (4.3) and (4.4) it follows that

$$\limsup_{n\to+\infty} \langle A(u_n), u_n-u\rangle \leqslant 0,$$

so

$$u_n \longrightarrow u \quad \text{in } W^{1,p}(\Omega)$$

(see Proposition 2.2). Then the continuity of the operator V implies that

$$V(u_n) \longrightarrow V(u) \quad \text{in } W^{1,p}(\Omega)^*,$$

so

$$u^* = V(u)$$

(see (4.2)). Also, we have

$$\langle V(u)_n, u_n\rangle \longrightarrow \langle V(u), u\rangle,$$

so V is pseudomonotone.

For every $u \in W^{1,p}(\Omega)$, we have

$$\begin{aligned}\langle V(u), u\rangle &= \gamma(u) + \|u\|_p^p - \int_\Omega \widehat{f}(z,\tau(u),D\tau(u))u\,dz \\ &\geqslant \|u\|^p - c_2\|u\|,\end{aligned}$$

for some $c_2 > 0$ (see (4.1)), so V is coercive.

We know that a pseudomonotone, coercive operator is surjective (see Gasiński-Papageorgiou [5, p. 336]). So, we can find $\widehat{u} \in W^{1,p}(\Omega)$ such that

$$V(\widehat{u}) = 0,$$

so

$$\begin{aligned}&\langle A(\widehat{u}), h\rangle + \int_\Omega |\widehat{u}|^{p-2}\widehat{u}h\,dz + \int_{\partial\Omega} \beta(z)|\widehat{u}|^{p-2}\widehat{u}h\,d\sigma \\ = &\int_\Omega \big(f(z,\tau(\widehat{u}),D\tau(\widehat{u})) + |\tau(\widehat{u})|^{p-2}\tau(\widehat{u})\big)h\,dz \quad \forall h \in W^{1,p}(\Omega). \quad (4.5)\end{aligned}$$

In (4.5) first we choose $h = (\widetilde{u} - \widehat{u})^+ \in W^{1,p}(\Omega)$. Then

$$\begin{aligned} & \langle A(\widehat{u}), (\widetilde{u} - \widehat{u})^+ \rangle + \int_\Omega |\widehat{u}|^{p-2}\widehat{u}(\widetilde{u} - \widehat{u})^+ \, dz + \int_{\partial\Omega} \beta(z)|\widehat{u}|^{p-2}\widehat{u}(\widetilde{u} - \widehat{u})^+ \, d\sigma \\ = & \int_\Omega \left(f(z, \widetilde{u}, D\widetilde{u}) + \widetilde{u}^{p-1}\right)(\widetilde{u} - \widehat{u})^+ \, dz \\ \geqslant & \langle A(\widetilde{u}), (\widetilde{u} - \widehat{u})^+ \rangle + \int_\Omega \widetilde{u}^{p-1}(\widetilde{u} - \widehat{u})^+ \, dz + \int_{\partial\Omega} \beta(z)\widetilde{u}^{p-1}(\widetilde{u} - \widehat{u})^+ \, d\sigma \end{aligned}$$

(from (3.9) and Green's identity), so

$$\widetilde{u} \leqslant \widehat{u}.$$

Next in (4.5) we choose $h = (\widehat{u} - c_+)^+ \in W^{1,p}(\Omega)$. Then

$$\begin{aligned} & \langle A(\widehat{u}), (\widehat{u} - c_+)^+ \rangle + \int_\Omega \widehat{u}^{p-1}(\widehat{u} - c_+)^+ \, dz + \int_{\partial\Omega} \beta(z)\widehat{u}^{p-1}(\widehat{u} - c_+)^+ \, d\sigma \\ = & \int_\Omega \left(f(z, c_+, 0) + c_+^{p-1}\right)(\widehat{u} - c_+)^+ \, dz \\ \leqslant & \langle A(c_+), (\widehat{u} - c_+)^+ \rangle + \int_\Omega c_+^{p-1}(\widehat{u} - c_+)^+ \, dz + \int_{\partial\Omega} \beta(z)c_+^{p-1}(\widehat{u} - c_+)^+ \, d\sigma \end{aligned}$$

(see hypotheses $H_1(i)$, H_0), so

$$\widehat{u} \leqslant c_+.$$

So, we have proved that

$$\widehat{u} \in [\widetilde{u}, c_+], \tag{4.6}$$

where $[\widetilde{u}, c_+] = \{y \in W^{1,p}(\Omega) : \ \widetilde{u}(z) \leqslant y(z) \leqslant c_+ \text{ for a.a. } z \in \Omega\}$. From (4.5), (4.6) and (4.1) we obtain that

$$\langle A(\widehat{u}), h \rangle + \int_{\partial\Omega} \beta(z)\widehat{u}^{p-1} h \, d\sigma = \int_\Omega f(z, \widehat{u}, D\widehat{u}) h \, dz \quad \forall h \in W^{1,p}(\Omega),$$

so $\widehat{u}$ is a positive solution of (1.1). As before the nonlinear regularity theory and the nonlinear maximum principle imply that $\widehat{u} \in \mathrm{int}C_+$ and as in Section 3 (see (3.10)), using the strong comparison principle (see Papageorgiou-Rădulescu-Repovš [10]), we have that $\widehat{u}(z) < c_+$ for all $z \in \overline{\Omega}$.

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