On Rational Functions Related to Algorithms for a Computation of Roots. I

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Article history:
Received 20 December 2019
Received in revised form
27 December 2019
Accepted 27 December 2019
Available online 31 December 2019

Abstract
We discuss a less known but surprising fact: a very old algorithm for computing square root known as the Bhaskara-Brouncker algorithm contains another and faster algorithms. A similar approach was obtained earlier by A.K. Yeyios [8] in 1992. By the way, we shall present a few useful facts as an essential completion of [8]. In particular, we present a direct proof that $k-th$ Yeyios iterative algorithm is of order k. We also observe that Chebyshev polynomials T_n and U_n are a special case of a more general construction. The most valuable idea followed this paper is contained in applications of a simple rational function $\Phi(w,z) = \frac{z-w}{z+w}$.

Key words: Algorithms, iterative methods, polynomials, recurrence relations

1. Introduction.

Bhaskara-Brouncker algorithm. Let $x_a[n] = \frac{p_n(a)}{q_n(a)}$, where

$$\begin{cases} p_{n+1}(a) = p_n(a) + q_n(a)a; \\ q_{n+1}(a) = p_n(a) + q_n(a); \\ p_1(a) = q_1(a) = 1. \end{cases}.$$

Thus $x_a[n+1] = \frac{x_a[n]+a}{x_a[n]+1}$. First nine elements of the sequel $x_a[n]$ are the following

$$x_a[1] = 1,\ x_a[2] = \frac{a+1}{2},\ x_a[3] = \frac{3a+1}{a+3},\ x_a[4] = \frac{a^2+6a+1}{4a+4},$$
$$x_a[5] = \frac{5a^2+10a+1}{a^2+10a+5},\ x_a[6] = \frac{a^3+15a^2+15a+1}{6a^2+20a+6},\ x_a[7] = \frac{7a^3+35a^2+21a+1}{a^3+21a^2+35a+7},$$
$$x_a[8] = \frac{a^4+28a^3+70a^2+28a+1}{8a^3+56a^2+56a+8},\ x_a[9] = \frac{9a^4+84a^3+126a^2+36a+1}{a^4+36a^3+126a^2+84a+9}.$$

There is known that $\lim_{n\to\infty} x_a[n] = \sqrt{a}$ and

$$\left|x_a[n] - \sqrt{a}\right| = \left|\frac{p_n(a)}{q_n(a)} - \sqrt{a}\right| \le \frac{1}{q_n(a)(p_n(a)+q_n(a)\sqrt{a})} < \frac{1}{2q_n(a)^2}.$$

Heron's algorithm.

$$y_a[n+1] = \frac{1}{2}\left(y_a[n] + \frac{a}{y_a[n]}\right),\ y_a[0] = 1.$$

2010 *Mathematics Subject Classification.* Primary 31C10 Secondary 32U35, 41A17.
Key words and phrases. Square roots, Heron algorithm, rational approximation.

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$$y_a[0]=1, y_a[1]=\frac{a+1}{2},\ y_a[2]=\frac{a^2+6a+1}{4a+4},\ y_a[3]=\frac{a^4+28a^3+70a^2+28a+1}{8a^3+56a^2+56a+8}.$$

If $F_a[n]=\frac{y_a[n]-\sqrt{a}}{y_a[n]+\sqrt{a}}$ then, as it was observed by [6] $F_a[n+1]=F_a[n]^2$, which implies

$$F_a[n]=F_a[0]^{2^n}=\left(\frac{y_a[0]-\sqrt{a}}{y_a[0]+\sqrt{a}}\right)^{2^n}.$$

If we put $\varepsilon_a[n]=y_a[n]-\sqrt{a}$ then

$$F_a[n]=\frac{\varepsilon_a[n]}{\varepsilon_a[n]+2\sqrt{a}},\ \varepsilon_a[n]=2\sqrt{a}\frac{F_a[n]}{1-F_a[n]}$$

and therefore

$$|\varepsilon_a[n+1]|=\frac{\varepsilon_a[n]^2}{|2\varepsilon_a[n]+2\sqrt{a}|}$$

with

$$\lim_{n\to\infty}\frac{|\varepsilon_a[n+1]|}{\varepsilon_a[n]^2}=\frac{1}{2\sqrt{a}}.$$

Halley's algorithm.

$$z_a[n+1]=\frac{z_a[n](3a+z_a[n]^2)}{3z_a[n]^2+a},\ z_a[0]=1.$$

$$z_a[1]=\frac{3a+1}{a+3},\ z_a[2]=\frac{9a^4+84a^3+126a^2+36a+1}{a^4+36a^3+126a^2+84a+9}.$$

Following [6] (who considered the case of Heron's algorithm) we also define in this case $G_a[n]=\frac{z_a[n]-\sqrt{a}}{z_a[n]+\sqrt{a}}$ and we easily check that $G_a[n+1]=G_a[n]^3$. Hence if $\varepsilon_a[n]=z_a[n]-\sqrt{a}$ we get

$$G_a[n]=\frac{\varepsilon_a[n]}{\varepsilon_a[n]+2\sqrt{a}},\ \varepsilon_a[n]=2\sqrt{a}\frac{G_a[n]}{1-G_a[n]}$$

and thus

$$|\varepsilon_a[n+1]|=|\varepsilon_a[n]|^3\frac{1}{4a}\frac{(1-G_a[n])^2}{1+G_a[n]+G_a[n]^2}$$

$$=|\varepsilon_a[n]|^3\frac{1}{3\varepsilon_a[n]^2+6\sqrt{a}\varepsilon_a[n]+4a}=|\varepsilon_a[n]|^3\frac{1}{3(\varepsilon_a[n]+\sqrt{a})^2+a}$$

with

$$\lim_{n\to\infty}\frac{|\varepsilon_a[n+1]|}{|\varepsilon_a[n]|^3}=\frac{1}{4a}.$$

Let us observe that

$$y_a[1]=x_a[2],\ y_a[2]=x_a[4],\ y_a[3]=x_a[8]$$

and

$$z_a[1]=x_2[3],\ z_a[2]=x_a[9].$$

We can suppose that

$$y_a[n] = x_a[2^n],\ z_a[n] = x_a[3^n],\ n = 0, 1, 2, \ldots$$

We shall check in the next section that it is really true.

Let $\Phi(w,z) = \frac{z-w}{z+w}$, $R_2(w,z) = \frac{1}{2}\left(z + \frac{w^2}{z}\right)$, $R_3(w,z) = \frac{z(3w^2+z^2)}{3z^2+w^2}$. Then

$$\Phi(w, R_2(w,z)) = \Phi(w,z)^2,\ \Phi(w, R_3(w,z)) = \Phi(w,z)^3.$$

The above observations for $F_a[n]$ and $G_a[n]$ are equivalent to $y_a[n+1] = R_2(\sqrt{a}, y_a[n])$, $z_a[n+1] = R_3(\sqrt{a}, z_a[n])$ and

$$\Phi(\sqrt{a}, R_2(\sqrt{a}, y_a[n])) = \Phi(\sqrt{a}, y_a[n])^2,\ \Phi(\sqrt{a}, R_3(\sqrt{a}, z_a[n])) = \Phi(\sqrt{a}, z_a[n])^3.$$

Now we introduce a sequence of rational functions $R_k(w,z)$ by

$$\Phi(w, R_k(w,z)) = \Phi(w,z)^k,\ k = 1, 2, \ldots$$

The basic properties of R_k, which play a crucial role, are contained in the following.

Theorem 1.1. *For all $n, m \geq 1$ we have $R_{mn}(w,z) = R_m(w, R_n(w,z))$. If we put for a fixed w $\Phi_w(z) = \Phi(w,z)$ then $\Phi_w^{-1}(z) = w\frac{1+z}{1-z}$ and*

$$R_k(w,z) = \Phi_w^{-1}(\Phi_w(z)^k) = w\frac{(z+w)^k + (z-w)^k}{(z+w)^k - (z-w)^k},\ k = 1, 2, \ldots$$

Let a sequence $\zeta_w[n]$ be defined by the recurrence formula $\zeta_w[n+1] = R_k(w, \zeta_w[n])$, $\zeta_w[0] = 1$. Then, if we consider the error function $E(w,z) = z - w$, we get

$$\frac{|E(w, \zeta_w[n+1])|}{|E(w, \zeta_w[n])|^k} = \frac{|E(w, \zeta_w[n+1]) + 2w|}{|E(w, \zeta_w[n]) + 2w|^k}.$$

Hence, if $\lim_{n\to\infty} E(w, \zeta_w[n]) = 0$, then

$$\lim_{n\to\infty} \frac{|E(w, \zeta_w[n+1])|}{|E(w, \zeta_w[n])|^k} = \left(\frac{1}{2|w|}\right)^{k-1},\ k \geq 2.$$

This means (cf. [7] *and* [8] *for the definition) that the iterative method $\zeta_w[n]$ is of order k.*

Proof. We have

$$\Phi(w, R_m(w, R_n(w,z)) = \Phi(w, R_n(w,z))^m = \Phi(w,z)^{nm} = \Phi(w, R_{mn}(w,z),$$

which gives $R_{mn}(w,z) = R_m(w, R_n(w,z))$.

□

2. Bhaskara-Brouncker algorithm gives Heron's and Halley's algorithmic sequences.

$$\begin{cases} p_0(a) = q_0(a) = 1, \; p_1(a) = a+1, \; q_1(a) = 2 \\ p_{n+1}(a) = p_n(a) + aq_n(a) \\ q_{n+1}(a) = p_n(a) + q_n(a) \end{cases}$$

$$\begin{cases} p_0(a) = q_0(a) = 1, \; p_1(a) = a+1, \; q_1(a) = 2 \\ p_{n+2}(a) = 2p_{n+1}(a) + (a-1)p_n(a) \\ q_{n+2}(a) = 2q_{n+1}(a) + (a-1)q_n(a) \end{cases}$$

If $P_n(a) = \frac{p_{n+1}(a)}{p_n(a)}, \; Q_n(a) = \frac{q_{n+1}(a)}{q_n(a)}$, then

$$P_{n+1}(a) = 1 + (a-1)/P_n(a), \; Q_{n+1}(a) = 1 + (a-1)/Q_n(a).$$

$$\begin{bmatrix} p_{n+1}(a) & p_n(a) \\ p_n(a) & p_{n-1}(a) \end{bmatrix} = \begin{bmatrix} a+1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ a-1 & 0 \end{bmatrix}^n, \; n = 0,1,2,3,\dots$$

$$\begin{bmatrix} q_{n+1}(a) & q_n(a) \\ q_n(a) & q_{n-1}(a) \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ a-1 & 0 \end{bmatrix}^n, \; n = 0,1,2,3,\dots$$

$$\frac{1}{a}\begin{bmatrix} 1 & -1 \\ -1 & a+1 \end{bmatrix} \begin{bmatrix} p_{n+1}(a) & p_n(a) \\ p_n(a) & p_{n-1}(a) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} q_{n+1}(a) & q_n(a) \\ q_n(a) & q_{n-1}(a) \end{bmatrix},$$

in particular

$$q_n(a) = \frac{p_{n+1}(a) - p_n(a)}{a}, \; \frac{p_n(a)}{q_n(a)} = \frac{ap_n(a)}{p_{n+1}(a) - p_n(a)} = \frac{a}{\frac{p_{n+1}(a)}{p_n(a)} - 1}.$$

Since

$$\begin{bmatrix} 2 & 1 \\ a-1 & 0 \end{bmatrix} = \begin{bmatrix} -\frac{1}{\sqrt{a}+1} & \frac{1}{\sqrt{a}-1} \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1-\sqrt{a} & 0 \\ 0 & 1+\sqrt{a} \end{bmatrix} \begin{bmatrix} \frac{1-a}{2\sqrt{a}} & \frac{1}{2}(1+\frac{1}{\sqrt{a}}) \\ \frac{-1+a}{2\sqrt{a}} & \frac{1}{2}(1-\frac{1}{\sqrt{a}}) \end{bmatrix}$$

and

$$\begin{bmatrix} 2 & 1 \\ a-1 & 0 \end{bmatrix}^n = \begin{bmatrix} -\frac{1}{\sqrt{a}+1} & \frac{1}{\sqrt{a}-1} \\ 1 & 1 \end{bmatrix} \begin{bmatrix} (1-\sqrt{a})^n & 0 \\ 0 & (1+\sqrt{a})^n \end{bmatrix} \begin{bmatrix} \frac{1-a}{2\sqrt{a}} & \frac{1}{2}(1+\frac{1}{\sqrt{a}}) \\ \frac{-1+a}{2\sqrt{a}} & \frac{1}{2}(1-\frac{1}{\sqrt{a}}) \end{bmatrix},$$

we get

$$p_n(a) = \frac{1}{2}\left((1+\sqrt{a})^{n+1} + (1-\sqrt{a})^{n+1}\right), \; n = 0,1,\dots$$

$$q_n(a) = \frac{1}{2\sqrt{a}}\left((1+\sqrt{a})^{n+1} - (1-\sqrt{a})^{n+1}\right), \; n = 0,1,\dots$$

Thus we have checked the following fact.

Proposition 2.1.

$$x_a[n] = \sqrt{a}\frac{(1+\sqrt{a})^n + (1-\sqrt{a})^n}{(1+\sqrt{a})^n - (1-\sqrt{a})^n}, \; n = 1,2,\dots$$

The above Proposition is also a Corollary to the following nice fact.

Theorem 2.2. *If* $x_a[n+1] = \frac{x_a[n]+a}{x_a[n]+1}$ *then*

$$\Phi(\sqrt{a}, x_a[n+1]) = g(\sqrt{a})\Phi(\sqrt{a}, x_a[n]), \tag{2.1}$$

where $g(u) = \frac{1-u}{1+u}$, $u \neq 1$.

Proof. Let $h(w,z) = \frac{z+w^2}{z+1}$. To check (2.1), we prove that $\Phi(w, h(w,z)) = g(w)\Phi(w,z)$. Really,

$$\Phi(w, h(w,z)) = \frac{\frac{z+w^2}{z+1} - w}{\frac{z+w^2}{z+1} + w} = \frac{z(1-w) - w(1-w)}{z(1+w) + w(1+w)} = g(w)\Phi(w,z).$$

□

Corollary 2.3. *For any positive integer* n *we have*

$$\Phi(\sqrt{a}, x_a[n]) = \frac{1-\sqrt{a}}{1+\sqrt{a}}\Phi(\sqrt{a}, x_a[n-1]) \tag{2.2}$$

$$= \left(\frac{1-\sqrt{a}}{1+\sqrt{a}}\right)^k \Phi(\sqrt{a}, x_a[n-k]) = \left(\frac{1-\sqrt{a}}{1+\sqrt{a}}\right)^n.$$

$$x_a[n] = \Phi^{-1}_{\sqrt{a}}\left(\left(\frac{1-\sqrt{a}}{1+\sqrt{a}}\right)^n\right) = \sqrt{a}\frac{(1+\sqrt{a})^n + (1-\sqrt{a})^n}{(1+\sqrt{a})^n - (1-\sqrt{a})^n}. \tag{2.3}$$

Proposition 2.4.

$$x_a[2n] = \frac{1}{2}(x_a[n] + a/x_a[n]), \; n = 1, 2, \dots$$

In particular,

$$x_a[2^{n+1}] = \frac{1}{2}(x_a[2^n] + a/x_a[2^n]), \; n = 0, 1, \dots$$

Proof.

$$\frac{1}{2}(x_a[n]+a/x_a[n]) = \frac{1}{2}\sqrt{a}\left(\frac{(1+\sqrt{a})^n + (1-\sqrt{a})^n}{(1+\sqrt{a})^n - (1-\sqrt{a})^n} + \frac{(1+\sqrt{a})^n - (1-\sqrt{a})^n}{(1+\sqrt{a})^n + (1-\sqrt{a})^n}\right)$$

$$= \sqrt{a}\frac{(1+\sqrt{a})^{2n} + (1-\sqrt{a})^{2n}}{(1+\sqrt{a})^{2n} - (1-\sqrt{a})^{2n}} = x_a[2n].$$

□

On can check, in a similar way, next proposition.

Proposition 2.5.

$$x_a[3n] = \frac{x_a[n](3a + x_a[n]^2)}{3x_a[n]^2 + a}, \; n = 1, 2, \dots$$

In particular,

$$x_a[3^{n+1}] = \frac{x_a[3^n](3a + x_a[3^n]^2)}{3x_a[3^n]^2 + a}, \; n = 0, 1, \dots$$

Define

$$h_a(t) = \frac{a+t}{1+z},\ h_a^2(t) = h_a(h_a(t)) = \frac{a + \frac{a+1}{2}t}{\frac{a+1}{2} + t},\ h_a^3(t) = \frac{a + \frac{3a+1}{a+3}t}{\frac{3a+1}{a+3} + t}, \dots$$

We propose to the reader to check following facts.

Proposition 2.6.

$$h_a^n(t) = \frac{a + x_a[n]t}{x_a[n] + t},\ n \geq 1,\ h_a^n(1) = x_a[n+1].$$

Corollary 2.7.

$$\lim_{n\to\infty} h_a^n(t) = \frac{a + \sqrt{a}t}{\sqrt{a} + t} = \sqrt{a}$$

and

$$\Phi(\sqrt{a}, h_a^n[t]) = \Phi(\sqrt{a}, t)\Phi(\sqrt{a}, x_a[n]).$$

3. Generalizations of Yeyios polynomials.

Yeyios [8] introduced polynomials P_n and Q_n in the following way.

$$\begin{cases} P_{n+1}(x) = xP_n(x) + Q_n(x)a; \\ Q_{n+1}(x) = P_n(x) + xQ_n(x); \\ P_0(x) = x,\ Q_0(x) = 1. \end{cases} .$$

$$\begin{cases} P_0(x) = x,\ Q_0(x) = 1,\ P_1(x) = x^2 + a,\ Q_1(x) = 2x,\ P_{-1}(x) = 1,\ Q_{-1}(x) = 0; \\ P_{n+2}(x) = 2xP_{n+1}(x) + (a - x^2)P_n(x); \\ Q_{n+2}(x) = 2xQ_{n+1}(x) + (a - x^2)Q_n(x). \end{cases}$$

$$\begin{bmatrix} P_{n+1}(x) & P_n(x) \\ P_n(x) & P_{n-1}(x) \end{bmatrix} = \begin{bmatrix} x^2 + a & x \\ x & 1 \end{bmatrix} \begin{bmatrix} 2x & 1 \\ a - x^2 & 0 \end{bmatrix}^n,\ n = 0, 1, 2, 3, \dots$$

$$\begin{bmatrix} Q_{n+1}(x) & Q_n(x) \\ Q_n(x) & Q_{n-1}(x) \end{bmatrix} = \begin{bmatrix} 2x & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 2x & 1 \\ a - x^2 & 0 \end{bmatrix}^n,\ n = 0, 1, 2, 3, \dots$$

$$\frac{1}{a} \begin{bmatrix} 1 & -x \\ -x & a + x^2 \end{bmatrix} \begin{bmatrix} P_{n+1}(x) & P_n(x) \\ P_n(x) & P_{n-1}(x) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & -2x \end{bmatrix} \begin{bmatrix} Q_{n+1}(x) & Q_n(x) \\ Q_n(x) & Q_{n-1}(x) \end{bmatrix}.$$

In particular

$$Q_n(x) = \frac{P_{n+1}(x) - xP_n(x)}{a},\ \frac{P_n(x)}{Q_n(x)} = \frac{aP_n(x)}{P_{n+1}(x) - xP_n(x)} = \frac{a}{\frac{P_{n+1}(x)}{P_n(x)} - x}.$$

Since

$$\begin{bmatrix} 2x & 1 \\ a - x^2 & 0 \end{bmatrix} = \begin{bmatrix} -\frac{1}{\sqrt{a}+x} \frac{1}{\sqrt{a}-x} & \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x - \sqrt{a} & 0 \\ 0 & x + \sqrt{a} \end{bmatrix} \begin{bmatrix} \frac{x^2-a}{2\sqrt{a}} & \frac{1}{2}(1 + \frac{x}{\sqrt{a}}) \\ \frac{-x^2+a}{2\sqrt{a}} & \frac{1}{2}(1 - \frac{x}{\sqrt{a}}) \end{bmatrix}$$

and

$$\begin{bmatrix} 2x & 1 \\ a-x^2 & 0 \end{bmatrix}^n = \begin{bmatrix} -\frac{1}{\sqrt{a}+x} \frac{1}{\sqrt{a}-x} & \\ 1 & 1 \end{bmatrix} \begin{bmatrix} (x-\sqrt{a})^n & 0 \\ 0 & (x+\sqrt{a})^n \end{bmatrix} \begin{bmatrix} \frac{x^2-a}{2\sqrt{a}} & \frac{1}{2}(1+\frac{x}{\sqrt{a}}) \\ \frac{-x^2+a}{2\sqrt{a}} & \frac{1}{2}(1-\frac{x}{\sqrt{a}}) \end{bmatrix},$$

we get

$$P_n(a,z) = \frac{1}{2}\left((z+\sqrt{a})^{n+1} + (z-\sqrt{a})^{n+1}\right), \; n = 0,1,\dots$$

$$Q_n(a,z) = \frac{1}{2\sqrt{a}}\left((z+\sqrt{a})^{n+1} - (z-\sqrt{a})^{n+1}\right), \; n = 0,1,\dots$$

$$S_n(a,z) = \frac{P_{n-1}(a,z)}{Q_{n-1}(a,z)} = \sqrt{a}\frac{(z+\sqrt{a})^n + (z-\sqrt{a})^n}{(z+\sqrt{a})^n - (z-\sqrt{a})^n}.$$

Proposition 3.1.

$$S_n(a,z) = zx_n[a/z^2].$$

Theorem 3.2. *(A.K. Yeyios [8]) For all $n, m \geq 1$*

$$S_{nm}(,az) = S_n(a, S_m(a,z)).$$

$$\hat{P}_n(w,z) = \frac{1}{2}\left((z+w)^n + (z-w)^n\right), \; \hat{Q}_n(z,w) = \frac{1}{2w}\left((z+w)^{n+1} - (z-w)^{n+1}\right),$$

$$\hat{S}_n(w,z) = \frac{\hat{P}_n(w,z)}{\hat{Q}_{n-1}(w,z)} = R_n(w,z).$$

We left to the reader to check the following properties.

Theorem 3.3. *For arbitrary $n, m \geq 1$*

- $\hat{Q}_n(w,z) = \frac{1}{w(n+2)}\frac{\partial \hat{P}_{n+2}(w,z)}{\partial w}$
- $\hat{S}_{nm}(w,z) = \hat{S}_n(w, \hat{S}_m(w,z))$.
- $\hat{P}_n(\sqrt{a},z) = P_{n-1}(a,z),\; \hat{Q}_n(\sqrt{a},z) = Q_n(a,z),\; \hat{S}_n(\sqrt{a},z) = S_n(a,z)$.
- $\hat{P}_n(\sqrt{z^2-1},z) = P_{n-1}(z^2-1,z) = T_n(z)$,
  $\hat{Q}_n(\sqrt{z^2-1},z) = Q_n(z^2-1,z) = U_n(z)$.
  Here T_n and U_n denote classical Chebyshev polynomials of the first and the second kind, respectively.

Remark 3.4. Let polynomials P_n, $n \geq 0$, satisfy recurrence $P_{n+2}(x) = 2xP_{n+1}(x) - P_n(x)$, $n \geq 0$ and take $Q_n(x) = P_{n+1}(x)/P_n(x)$ which will give a relation $Q_{n+1}(x) = 2x - 1/Q_n(x)$.

Now calculate $\Phi(w, Q_{n+1}(x) - x)$:

$$\Phi(w, Q_{n+1}(x) - x) = \frac{(x-w)Q_n(x) - 1}{(x+w)Q_n(x) - 1} = \frac{x-w}{x+w}\frac{Q_n(x) - 1/(2x-w)}{Q_n(x) - 1/(2x+w)}$$

$$= \Phi(w,x)\frac{Q_n(x) - x + x - 1/(2x-w)}{Q_n(x) - x + x - 1/(2x+w)}.$$

If we choose w so that $x - 1/(2x-w) = -w,\ x - 1/(2x+w) = w$, we shall get

$$\Phi(w, Q_{n+1}(x) - x) = \Phi(w,x)\Phi(w, Q_n(x) - x).$$

A proper choice is $w = \sqrt{x^2-1}$ and then

$$\Phi(w, Q_{n+1}(x) - x) = \Phi(w,x)^k\Phi(w, Q_{n-k+1}(x) - x),\ k = 1, \dots, n+1.$$

Hence

$$\Phi(w, Q_n(x) - x) = \Phi(w,x)^n\Phi(w, Q_0(x) - x),$$

$$Q_n(x) - x = \Phi_w^{-1}\left(\Phi(w,x)^n\Phi(w,Q_0(x)-x)\right) = w\frac{1+\Phi(w,x)^n\Phi(w,Q_0(x)-x)}{1-\Phi(w,x)^n\Phi(w,Q_0(x)-x)},$$

$$Q_n(x) = x + w\frac{1+\Phi(w,x)^n\Phi(w,Q_0(x)-x)}{1-\Phi(w,x)^n\Phi(w,Q_0(x)-x)}$$

and we can obtain (in an alternative way) formulas for $P_n(x)$. Consider, as an example, $P_0(x) = 1, P_1(x) = x,\ Q_0(x) = x$. Then $\Phi(w, Q_0(x) - x) = -1$, whence

$$Q_n(x) = x + w\frac{1-\Phi(w,x)^n}{1+\Phi(w,x)^n} = \frac{x(1+\Phi(w,z)^n) + w(1-\Phi(w,z)^n)}{1+\Phi(w,z)^n}$$

$$= \frac{x + w + (x-w)\Phi(w,x)^n}{1+\Phi(w,x)^n} = (x+w)\frac{1+\Phi(w,x)^{n+1}}{1+\Phi(w,x)^n}.$$

Since $P_n(x) = Q_{n-1}(x)\cdot Q_{n-2}(x)\cdots Q_0(x)$, we get

$$P_n(x) = (x+w)^n\frac{1+\Phi(w,x)^n}{1+\Phi(w,x)^0} = \frac{1}{2}\left((x+w)^n + (x-w)^n\right)$$

which is known as a formula for Chebyshev polynomials $P_n(x) = T_n(x)$.

Acknowledgment. The author was partially supported by the National Science Centre, Poland, 2017/25/B/ST1/00906.

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