Chebyshev Polynomials and Continued Fractions Related

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Abstract
Let p, q be complex polynomials, $\deg p > \deg q \geq 0$. We consider the family of polynomials defined by the recurrence $P_{n+1} = 2pP_n - qP_{n-1}$ for $n = 1, 2, 3, \ldots$ with arbitrary P_1 and P_0 as well as the domain of the convergence of the infinite continued fraction

$$f(z) = 2p(z) - \cfrac{q(z)}{2p(z) - \cfrac{q(z)}{2p(z) - \ldots}}.$$

Key words: Chebyshev polynomials, continued fractions, Binet formula, Cassini identity

1 Some polynomials of the Chebyshev type

Let P_0 and P_1 be polynomials of one complex variable, $\deg P_1 > \deg P_0 \geq 0$. Let p, q be polynomials of one complex variable, $\deg p > \deg q \geq 0$, $q \neq 0$. Define the family of polynomials P_n by the recurrence formula

$$P_{n+1}(z) = 2p(z)P_n(z) - q(z)P_{n-1}(z), \quad n = 1,\ 2,\ 3, \ldots. \tag{1}$$

Note that (1) gives the Chebyshev polynomials of

- *the first kind* T_n for $P_0(z) = 1$, $P_1(z) = z$, $p(z) = z$ and $q(z) = 1$
- *the second kind* U_n for $P_0(z) = 1$, $P_1(z) = 2z$, $p(z) = z$ and $q(z) = 1$
- *the third kind* V_n for $P_0(z) = 1$, $P_1(z) = 2z - 1$, $p(z) = z$ and $q(z) = 1$
- *the fourth kind* W_n for $P_0(z) = 1$, $P_1(z) = 2z + 1$, $p(z) = z$ and $q(z) = 1$.

(See [2], Appendix B, Table B.2).

We write the recurrence (1) in the matrix form

$$\begin{bmatrix} P_{n+1} & P_n \\ P_n & P_{n-1} \end{bmatrix} = \begin{bmatrix} 2p & -q \\ 1 & 0 \end{bmatrix} \begin{bmatrix} P_n & P_{n-1} \\ P_{n-1} & P_{n-2} \end{bmatrix} \tag{2}$$

proceeding as in [1], p.80, where the Fibonacci sequence was considered, defined by the similar recurrence $\begin{bmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} F_n & F_{n-1} \\ F_{n-1} & F_{n-2} \end{bmatrix}$ with $F_0 = 0$ and $F_1 = 1$.

Note that the characteristic polynomial

$$w(\lambda) = \det \begin{bmatrix} 2p - \lambda & -q \\ 1 & -\lambda \end{bmatrix} = (\lambda - p)^2 - p^2 + q$$

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of the matrix

$$\begin{bmatrix} 2p & -q \\ 1 & 0 \end{bmatrix} \tag{3}$$

addmits two different roots

$$\lambda_1 = p + \sqrt{p^2 - q} \text{ and } \lambda_2 = p - \sqrt{p^2 - q}, \tag{4}$$

as the polynomial q is assumed to be nonzero.

Theorem 1.1 *For the polynomial P_n defined by* (1) *we get the following formula*

$$P_n = \frac{1}{\lambda_1 - \lambda_2}\left[\left(\lambda_1^n - \lambda_2^n\right)P_1 - \lambda_1\lambda_2\left(\lambda_1^{n-1} - \lambda_2^{n-1}\right)P_0\right] \tag{5}$$

where λ_1 and λ_2 are the eigenvalues (4) *of the matrix* (3).

Proof. By the Jordan decomposition of the matrix (3) we get

$$\begin{bmatrix} 2p & -q \\ 1 & 0 \end{bmatrix} = \frac{1}{\lambda_1 - \lambda_2}\begin{bmatrix} \lambda_1 & \lambda_2 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}\begin{bmatrix} 1 & -\lambda_2 \\ -1 & \lambda_1 \end{bmatrix}.$$

The n-th power of the matrix (3) equals

$$\begin{bmatrix} 2p & -q \\ 1 & 0 \end{bmatrix}^n = \frac{1}{\lambda_1 - \lambda_2}\begin{bmatrix} \lambda_1 & \lambda_2 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} \lambda_1^n & 0 \\ 0 & \lambda_2^n \end{bmatrix}\begin{bmatrix} 1 & -\lambda_2 \\ -1 & \lambda_1 \end{bmatrix}.$$

Hence, by the recurrence

$$\begin{aligned}\begin{bmatrix} P_{n+1} & P_n \\ P_n & P_{n-1} \end{bmatrix} &= \begin{bmatrix} 2p & -q \\ 1 & 0 \end{bmatrix}\begin{bmatrix} P_n & P_{n-1} \\ P_{n-1} & P_{n-2} \end{bmatrix} = \begin{bmatrix} 2p & -q \\ 1 & 0 \end{bmatrix}^2\begin{bmatrix} P_{n-1} & P_{n-2} \\ P_{n-2} & P_{n-3} \end{bmatrix} = \dots \\ &= \begin{bmatrix} 2p & -q \\ 1 & 0 \end{bmatrix}^{n-1}\begin{bmatrix} P_2 & P_1 \\ P_1 & P_0 \end{bmatrix}\end{aligned}$$

we obtain

$$\begin{bmatrix} P_{n+1} & P_n \\ P_n & P_{n-1} \end{bmatrix} = \frac{1}{\lambda_1 - \lambda_2}\begin{bmatrix} \lambda_1 & \lambda_2 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} \lambda_1^{n-1} & 0 \\ 0 & \lambda_2^{n-1} \end{bmatrix}\begin{bmatrix} 1 & -\lambda_2 \\ -1 & \lambda_1 \end{bmatrix}\begin{bmatrix} P_2 & P_1 \\ P_1 & P_0 \end{bmatrix}.$$

Multiplying the above matrices we get

$$P_n = \frac{1}{\lambda_1 - \lambda_2}\left[\left(\lambda_1^n - \lambda_2^n\right)P_1 - \lambda_1\lambda_2\left(\lambda_1^{n-1} - \lambda_2^{n-1}\right)P_0\right]$$

□

Note that (5) corresponds to the well known *Binet formula* for the Fibonacci sequence

$$F_n = \frac{\mu_1^n - \mu_2^n}{\mu_1 - \mu_2} = \frac{1}{\sqrt{5}}\left[\left(\frac{\sqrt{5}+1}{2}\right)^n - \left(\frac{-\sqrt{5}+1}{2}\right)^n\right]$$

where $\mu_1 = \frac{\sqrt{5}+1}{2}$ and $\mu_2 = \frac{-\sqrt{5}+1}{2}$ are eigenvalues of the matrix $\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$ that defines the Fibonacci sequence $\begin{bmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} F_n & F_{n-1} \\ F_{n-1} & F_{n-2} \end{bmatrix}$ with $F_0 = 0$ and $F_1 = 1$.

Remark 1.2 The formula (5) works well with two known formulae (see [2] 1.49 and 1.52) for the Chebyshev polynomials of the first kind $T_0(x) = 1$, $T_1(x) = x$, $T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x)$ if we put $p(x) = x$, $q(x) = 1$, $\lambda_1(x) = x + \sqrt{x^2-1}$, $\lambda_2(x) = x - \sqrt{x^2-1}$, $P_0(x) = 1$ and $P_1(x) = x$:

$$\begin{aligned} T_n(x) &= \frac{1}{\lambda_1 - \lambda_2}\Big[\big(\lambda_1^n - \lambda_2^n\big)x - \big(\lambda_1^{n-1} - \lambda_2^{n-1}\big)\Big] \\ &= \frac{1}{\lambda_1 - \lambda_2}\Big[\lambda_1^n\big(x - \frac{1}{\lambda_1}\big) - \lambda_2^n\big(x - \frac{1}{\lambda_2}\big)\Big] \\ &= \frac{1}{2}(\lambda_1^n + \lambda_2^n) \\ &= \frac{1}{2}\big((x + \sqrt{x^2-1})^n + (x - \sqrt{x^2-1})^n\big), \quad |x| \geq 1, \end{aligned}$$

and for the Chebyshev polynomials of the second kind $U_0(x) = 1$, $U_1(x) = 2x$, $U_{n+1}(x) = 2xU_n(x) - U_{n-1}(x)$ if we put $p(x) = x$, $q(x) = 1$, $\lambda_1(x) = x + \sqrt{x^2-1}$, $\lambda_2(x) = x - \sqrt{x^2-1}$, $P_0(x) = 1$ and $P_1(x) = 2x$:

$$\begin{aligned} U_n(x) &= \frac{1}{\lambda_1 - \lambda_2}\Big[\big(\lambda_1^n - \lambda_2^n\big)2x - \big(\lambda_1^{n-1} - \lambda_2^{n-1}\big)\Big] \\ &= \frac{1}{\lambda_1 - \lambda_2}\Big[\lambda_1^n\big(2x - \frac{1}{\lambda_1}\big) - \lambda_2^n\big(2x - \frac{1}{\lambda_2}\big)\Big] \\ &= \frac{1}{\lambda_1 - \lambda_2}(\lambda_1^{n+1} - \lambda_2^{n+1}) \\ &= \frac{1}{2\sqrt{x^2-1}}\big((x + \sqrt{x^2-1})^{n+1} - (x - \sqrt{x^2-1})^{n+1}\big), \quad |x| \geq 1. \end{aligned}$$

Proceeding as above we get the next two formulae for the Chebyshev polynomials of the third and the fourth kind V_n, W_n, respectively:

$$\begin{aligned} V_n(x) &= \frac{1}{\lambda_1 - \lambda_2}\Big[\big(\lambda_1^n - \lambda_2^n\big)(2x-1) - \big(\lambda_1^{n-1} - \lambda_2^{n-1}\big)\Big] \\ &= \frac{1}{\lambda_1 - \lambda_2}\Big[\lambda_1^n\big(2x - 1 - \frac{1}{\lambda_1}\big) - \lambda_2^n\big(2x - 1 - \frac{1}{\lambda_2}\big)\Big] \\ &= \frac{1}{\lambda_1 - \lambda_2}(\lambda_1^{n+1} - \lambda_2^{n+1}) - \frac{1}{\lambda_1 - \lambda_2}(\lambda_1^n - \lambda_2^n) \\ &= U_n(x) - U_{n-1}(x) \end{aligned}$$

and

$$W_n(x) = \frac{1}{\lambda_1-\lambda_2}\Big[\big(\lambda_1^n-\lambda_2^n\big)(2x+1)-\big(\lambda_1^{n-1}-\lambda_2^{n-1}\big)\Big]$$
$$= \frac{1}{\lambda_1-\lambda_2}\Big[\lambda_1^n\big(2x+1-\frac{1}{\lambda_1}\big)-\lambda_2^n\big(2x+1-\frac{1}{\lambda_2}\big)\Big]$$
$$= \frac{1}{\lambda_1-\lambda_2}(\lambda_1^{n+1}-\lambda_2^{n+1})+\frac{1}{\lambda_1-\lambda_2}(\lambda_1^n-\lambda_2^n)$$
$$= U_n(x)+U_{n-1}(x)$$

Remark 1.3 If $\deg q = 0$, i.e. q is a nonzero constant, one may continue defining polynomials P_n for negative integers putting initial polynomials P_0, P_1 arbitrarily and the recurrence formula $P_{n-1} = -\frac{1}{q}P_{n+1}+\frac{2p}{q}P_n$ equivalent to the relation $P_{n+1} = 2pP_n(z)-qP_{n-1}$. Same as before we have

$$\begin{bmatrix} P_n & P_{n-1} \\ P_{n-1} & P_{n-2} \end{bmatrix} = \begin{bmatrix} 2p & -q \\ 1 & 0 \end{bmatrix}^{-1} \begin{bmatrix} P_{n+1} & P_n \\ P_n & P_{n-1} \end{bmatrix}$$

as

$$\begin{bmatrix} P_{n+1} & P_n \\ P_n & P_{n-1} \end{bmatrix} = \begin{bmatrix} 2p & -q \\ 1 & 0 \end{bmatrix} \begin{bmatrix} P_{n+1} & P_n \\ P_n & P_{n-1} \end{bmatrix}$$

We obtain

$$\begin{bmatrix} P_0 & P_{-1} \\ P_{-1} & P_{-2} \end{bmatrix} = \begin{bmatrix} 2p & -q \\ 1 & 0 \end{bmatrix}^{-1} \begin{bmatrix} P_1 & P_0 \\ P_0 & P_{-1} \end{bmatrix}$$

and

$$\begin{bmatrix} P_{-n+1} & P_{-n} \\ P_{-n} & P_{-n-1} \end{bmatrix} = \begin{bmatrix} 2p & -q \\ 1 & 0 \end{bmatrix}^{-n} \begin{bmatrix} P_1 & P_0 \\ P_0 & P_{-1} \end{bmatrix}$$
$$= \frac{1}{\lambda_1-\lambda_2}\begin{bmatrix} \lambda_1 & \lambda_2 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} \lambda_1^{-n} & 0 \\ 0 & \lambda_2^{-n} \end{bmatrix}\begin{bmatrix} 1 & -\lambda_2 \\ -1 & \lambda_1 \end{bmatrix}\begin{bmatrix} P_1 & P_0 \\ P_0 & P_{-1} \end{bmatrix}$$

Multiplying the above matrices we get an analogous formula as (5) in Theorem 1.1:

$$P_{-n} = \frac{1}{\lambda_1-\lambda_2}\Big[\big(\lambda_1^{-n}-\lambda_2^{-n}\big)P_1-\lambda_1\lambda_2\big(\lambda_1^{-n-1}-\lambda_2^{-n-1}\big)P_0\Big]$$

□

Calculating the determinant of the matrix

$$\begin{bmatrix} P_{n+1} & P_n \\ P_n & P_{n-1} \end{bmatrix} = \frac{1}{\lambda_1-\lambda_2}\begin{bmatrix} \lambda_1 & \lambda_2 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} \lambda_1^{n-1} & 0 \\ 0 & \lambda_2^{n-1} \end{bmatrix}\begin{bmatrix} 1 & -\lambda_2 \\ -1 & \lambda_1 \end{bmatrix}\begin{bmatrix} P_2 & P_1 \\ P_1 & P_0 \end{bmatrix}$$

we get *the Cassini type identity* for the polynomials P_n, corresponding to *the Cassini identity* for the Fibonacci sequence $F_{n+1}F_{n-1}-F_n^2 = \det\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^n = (-1)^n$:

$$(\lambda_1-\lambda_2)^2\det\begin{bmatrix} P_{n+1} & P_n \\ P_n & P_{n-1} \end{bmatrix} = \det\begin{bmatrix} \lambda_1 & \lambda_2 \\ 1 & 1 \end{bmatrix}\det\begin{bmatrix} \lambda_1^{n-1} & 0 \\ 0 & \lambda_2^{n-1} \end{bmatrix}\det\begin{bmatrix} 1 & -\lambda_2 \\ -1 & \lambda_1 \end{bmatrix}\det\begin{bmatrix} P_2 & P_1 \\ P_1 & P_0 \end{bmatrix}$$
$$\det\begin{bmatrix} P_{n+1} & P_n \\ P_n & P_{n-1} \end{bmatrix} = \det\begin{bmatrix} \lambda_1^{n-1} & 0 \\ 0 & \lambda_2^{n-1} \end{bmatrix}\det\begin{bmatrix} P_2 & P_1 \\ P_1 & P_0 \end{bmatrix}$$

Since $\lambda_1\lambda_2 = q$ we get the following remark.

Remark 1.4 The Cassini type identity for the polynomials P_n defined by (1) holds:

$$P_{n+1}P_{n-1} - P_n^2 = q^{n-1}(P_2P_0 - P_1^2)$$

which implies the four known formulae for the Chebyshev polynomials of the first, second, third and fourth kind, respectively:

$$\begin{array}{rrr} T_{n+1}(x)T_{n-1}(x) & -T_n^2(x) = & x^2-1 \\ U_{n+1}(x)U_{n-1}(x) & -U_n^2(x) = & -1 \\ V_{n+1}(x)V_{n-1}(x) & -V_n^2(x) = & -2x-2 \\ W_{n+1}(x)W_{n-1}(x) & -W_n^2(x) = & 2x-2. \end{array}$$

Theorem 1.5 *Let P_n be the sequence of polynomials defined by* (1)*. The quotient P_{n+1}/P_n converges uniformly on compact subsets of the set*

$$\{z \in \mathbb{C} : \left|\frac{\lambda_1(z)}{\lambda_2(z)}\right| > 1\}$$

to the limit λ_1. The limit does not depend on the initial polynomials P_0 and P_1.

Proof. By (5) the quotient of polynomials P_{n+1} and P_n equals

$$\begin{aligned} \frac{P_{n+1}}{P_n} &= \frac{\left(\lambda_1^{n+1} - \lambda_2^{n+1}\right)P_1 - \lambda_1\lambda_2\left(\lambda_1^n - \lambda_2^n\right)P_0}{\left(\lambda_1^n - \lambda_2^n\right)P_1 - \lambda_1\lambda_2\left(\lambda_1^{n-1} - \lambda_2^{n-1}\right)P_0} \\ &= \frac{\left(\lambda_1 - \lambda_2(\lambda_2/\lambda_1)^n\right)P_1 - \lambda_1\lambda_2\left(1 - (\lambda_2/\lambda_1)^n\right)P_0}{\left(1 - (\lambda_2/\lambda_1)^n\right)P_1 - \lambda_2\left(1 - (\lambda_2/\lambda_1)^{n-1}\right)P_0} \end{aligned}$$

It converges on compact subesets the set $\{z \in \mathbb{C} : \left|\frac{\lambda_1(z)}{\lambda_2(z)}\right| > 1\}$ uniformly to the limit

$$\frac{\lambda_1P_1 - \lambda_1\lambda_2P_0}{P_1 - \lambda_2P_0} = \lambda_1$$

that is independent of P_0 and P_1. □

2 Continued fractions related to polynomials P_n

Consider the infinite continued fraction

$$f(z) = 2p(z) - \cfrac{q(z)}{2p(z) - \cfrac{q(z)}{2p(z) - \ldots}}$$

and the rational functions r_k related to f:

$$
\begin{aligned}
r_1 &= P_1/P_0 \\
r_2 &= P_2/P_1 &&= \frac{2pP_1 - qP_0}{P_1} &&= 2p - \frac{q}{P_1/P_0} \\
r_3 &= P_3/P_2 &&= \frac{2pP_2 - qP_1}{P_2} &&= 2p - \frac{q}{P_2/P_1} \\
&\ldots\ldots &&\ldots\ldots &&\ldots \\
r_{n+1} &= P_{n+1}/P_n &&= \frac{2pP_n - qP_{n-1}}{P_{n-1}} &&= 2p - \frac{q}{P_n/P_{n-1}} \\
&\ldots\ldots &&\ldots\ldots &&\ldots
\end{aligned}
$$

It is easy to see that the function f is the limit of the sequence r_k.

Theorem 2.1 *For arbitrary polynomials p and q such that* $\deg p > \deg q > 0$, $q \neq 0$, *the continued fraction*

$$f(z) = 2p(z) - \frac{q(z)}{2p(z) - \frac{q(z)}{2p(z) - \ldots}}$$

is a holomorphic function on the set $\{z \in \mathbb{C} : \left|\frac{\lambda_1(z)}{\lambda_2(z)}\right| > 1\}$ *where λ_1 and λ_2 are eigenvalues* (4) *of the matrix* (3).

Proof. The statement follows from Theorem 1.5. □

Remark 2.2. In the simple case $p(z) = 2\alpha z$ and $q(z) = \beta^2$ the set $\{z \in \mathbb{C} : \left|\frac{\lambda_1(z)}{\lambda_2(z)}\right| > 1\}$ is the exterior of the interval connecting two points on the complex plane: $\frac{\beta}{\alpha}$ and $-\frac{\beta}{\alpha}$. See Figure 1 for the density plot of the absolute value of the function r_{60} for $p(z) = 2z$ and $q(z) = i = \left(\frac{1+i}{\sqrt{2}}\right)^2$ with a visible *scar* connecting the points $-\frac{1+i}{\sqrt{2}}$ and $\frac{1+i}{\sqrt{2}}$. The following plots exhibit the density plot of $|r_{60}|$ and more complex *scars* containing points where the sequence r_k is divergent if $p(z) = z^2 + \frac{i}{10}$, $q(z) = i$ (Figure 2), $p(z) = iz^3 + \frac{1}{2}$, $q(z) = -\frac{1}{3}$ (Figure 3), $p(z) = iz^2 - \frac{1}{5}z + \frac{1}{10}$, $q(z) = (i - \frac{1}{3})z + \frac{i}{7} + \frac{1}{5}$ (Figure 4).

The plots were created using *Mathematica* Wolfram Research program.

References

[1] Donald E. Knuth, *The Art of Computer Programming*, Addison Wesley, 2nd edition, 1973.

[2] John C. Mason, David Handscomb, *Chebyshev polynomials*, Chapman & Hall, 2003.

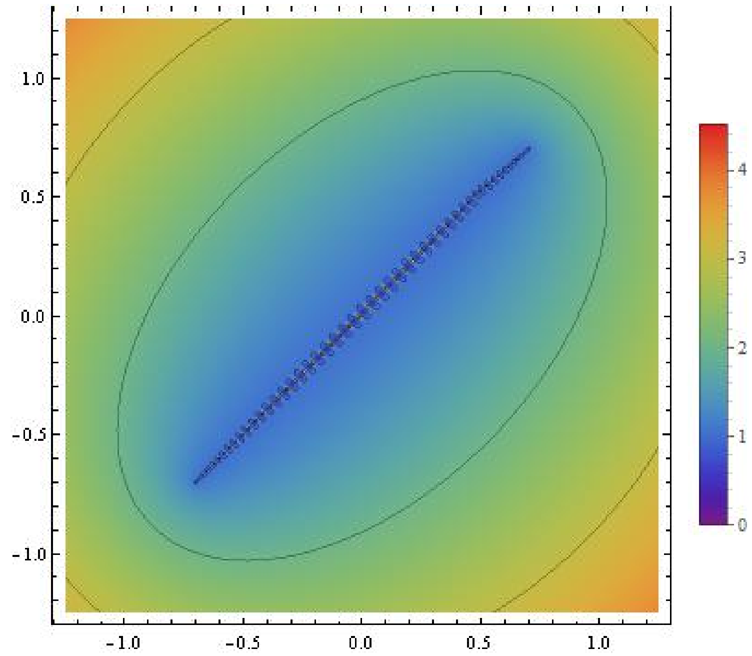

Figure 1: Density plot of $|r_{60}|$ for $p(z) = 2z$ and $q(z) = i$.

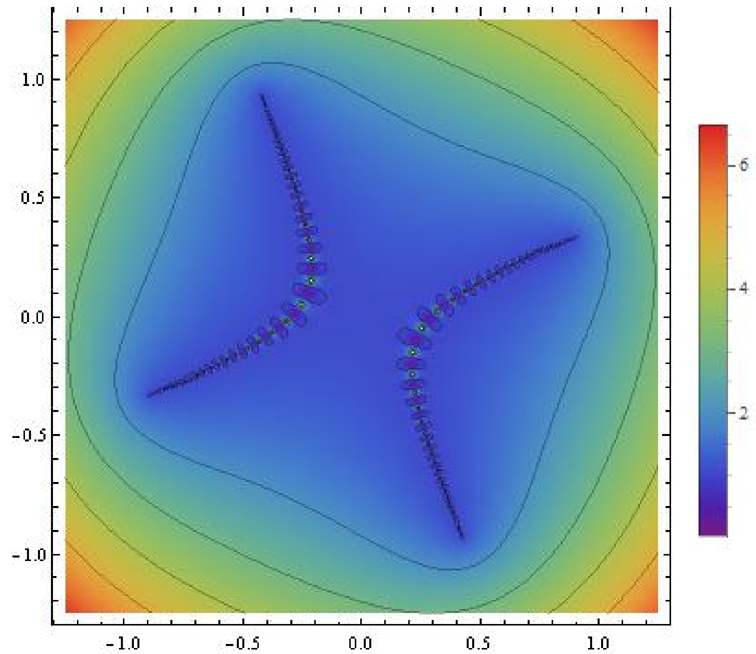

Figure 2: Density plot of $|r_{60}|$ for $p(z) = z^2 + \frac{i}{10}$, $q(z) = i$.

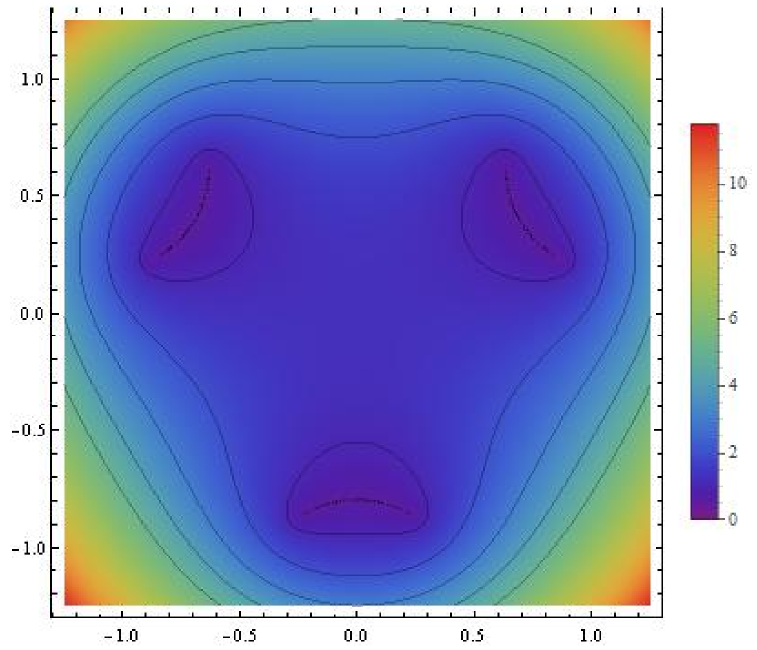

Figure 3: Density plot of $|r_{60}|$ for $p(z) = iz^3 + \frac{1}{2}$, $q(z) = -\frac{1}{3}$.

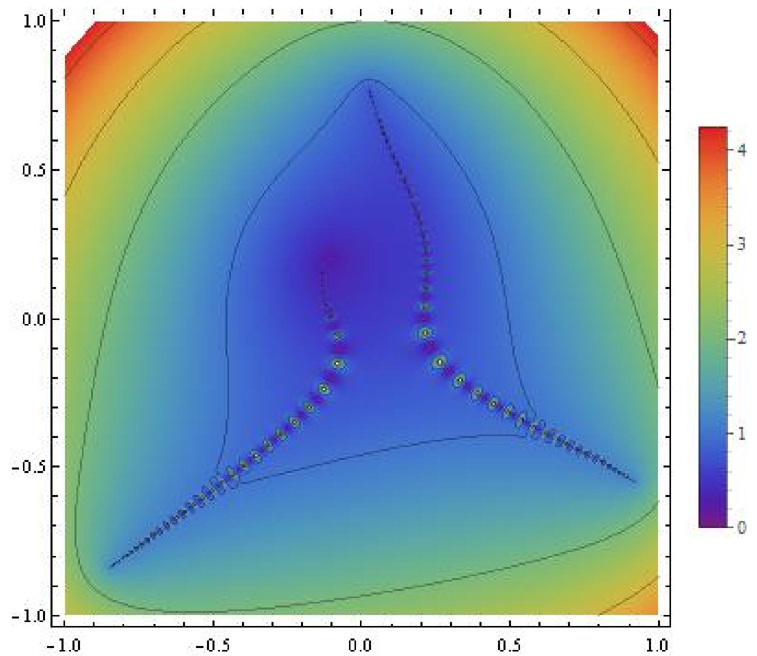

Figure 4: Density plot of $|r_{60}|$ for $p(z) = iz^2 - \frac{1}{5}z + \frac{1}{10}$, $q(z) = (i - \frac{1}{3})z + \frac{i}{7} + \frac{1}{5}$.