State estimators and observers for continuous and discrete linear systems. Part 2. Integral observers for exact state reconstruction

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Article history:
Received 5 May 2019
Received in revised form
12 July 2019
Accepted 12 July 2019
Available online 12 July 2019

Abstract
In the paper, the exact state observers will be presented. The state estimators and observers can be used in technical processes for many purposes like the fault detection and diagnosis, the implementation of the state controllers, and soft reconstruction of inaccessible for measurements variables of the system. As the standard, for continuous systems the differential estimators of Kalman filter or Luenberger type observer are commonly used. However, if the initial conditions of the real state are unknown, both estimators guarantee only an asymptotic quality of the real state tracking. The paper presents another type of the state observers, which for continuous system have the structure given by two integral operators. Based on measurements of the system input and output signals on some predefined finite time interval T, they can reconstruct the initial state exactly. In on-line version, the exact state reconstruction is performed continuously for every t, based on special procedure executed within two moving windows of width T, on sliding time interval [t-T, t].

Keywords: exact state observers, linear systems, state observers with minimal norm

Introduction

In many applications of the fault detection algorithms in technological processes, the model-based approach is used. Comparing the values of the measured output vector from the real process with the simulated values given by the mathematical model one can notice the differences. Based on these residuals one can concluded about the parameters faults, which has taken place in the process [1]. Sometimes the output vector $y(t) \in R^m$ does not contain enough information about the place where the fault occurred. Then for the exact fault isolation and identification the information about the full state vector values $x(t) \in R^n$ is needed, where n>m. In many cases, the linear time invariant models (LTI) with lumped parameters may approximate the dynamics of the real systems. The high order n of such a model is a consequence of the presence of n independent storage elements (for energy, mass or momentum) in the structure of the real system. The process state vector x(t) is very useful also for the control algorithms. Stabilization of technological process at the working point by the controllers based on the only output signal measurements y(t) is very often not enough. Hence, the knowledge of the state vector x(t) is fundamental in many tasks of the system analysis and synthesis. Unfortunately, in many cases the measurement of the entire state vector x(t) is not possible. Therefore, the state should be reconstructed (calculated) based on the mathematical model of the process, given by the matrix differential state equation and algebraic equation of the output. Such reconstruction is possible if the system is state observable. In the authors previous paper [2] the differential asymptotic state estimators were presented. Their structure was based on the differential equation. The main disadvantage of the Kalman Filter or Luenberger observer [3], [4] for state estimation is the asymptotic convergence of the state estimate to the real state value and lack of the information about the estimation error value. The power of modern computers makes possible application of quite different on-line observation algorithms. They reconstruct the exact value of the state vector, e.g. $x(t_0)$ at the moment t_0, making calculation on time interval $[t_0, t_0+T]$. The value of T may be fixed in advance. It is also possible reconstruction of the current state x(t) for any time t, making calculation on the past time interval [t-T, t]. One can see that in on-line mode, this observer forms moving observation window MWO (with receding horizon) and for the current time t, the calculations are based on finite history of measurement samples of the input u and output y signals. This property of the exact state information is especially valuable for the system fault detection,

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stabilization of the state and other critical real-time applications, where finite and the known time T of finishing the calculation task is very important. If the plant model is known and the input/output measurements have negligible level of the noise, then the exact observers can reconstruct (observe) the state, regardless of the lack of knowledge of the initial conditions of the system. Under disturbed measurements, the exact observers with the minimal norm are particularly useful. The structure of the exact state observers are based on integral operations and do not use the differential equation solution.

State observability condition in linear time invariant systems

We will recall the basic relationships that lead to the definition of observability. Let the continuous and linear model of homogeneous system be given,

$$\dot{x}(t) = A\,x(t), \qquad x(0) = x_0$$
$$y(t) = C\,x(t) \tag{1}$$

Where $x(t) \in R^n$, $u(t) \in R^r$ and $y(t) \in R^m$, for $\forall t \geq 0$. The initial state x(0) is unknown x(0)=? The output signal y(t) is measured and is known. Because the dimension m < n, the matrix C is rectangular (less equations than unknown variables). Hence based on single measurement of output vector $y(t_1)$, the state vector $x(t_1)$ cannot be calculated.

Standard formula for the output y(t) of the above LTI system is

$$y(t) = C\,e^{At}\,x(0) \tag{2}$$

Multiplying the both sides of (2) by transposition of the suitable matrix one can obtained

$$e^{A't}\,C'\,y(t) = e^{A't}\,C'\,C\,e^{At}\,x(0) \tag{3}$$

Obtained matrix $e^{A't}\,C'\,C\,e^{At}$ is square, however, still singular for any t.

Integration of (3) in interval [0, T] enables calculation of x(0) if and only if the square Gram matrix M_0 is non-singular and the history of the output signal y(t) on this interval is known.

$$x(0) = M_0^{-1} \int_0^T e^{A't}\,C'\,y(t)\,dt \tag{4}$$

where $$M_0 = \int_0^T e^{A't}\,C'\,C\,e^{At}\,dt \tag{5}$$

Nonsingularity of M_0 is the well-known necessary and sufficient condition for the system, to be state observable. The equivalent algebraic formula for the state observability has the form

$$rank\;Q_o = rank\begin{bmatrix} C \\ C\,A \\ \vdots \\ C\,A^{n-m} \end{bmatrix} = n, \qquad \text{for } 1 \leq m \leq n \tag{6}$$

From this equivalence, it is easy to see that for continuous systems the state observability does not depend on the time observation T.

The exact state reconstruction in finite time interval

The standard model of LTI system is given

$$\dot{x}(t) = Ax(t) + Bu(t), \qquad x(0) = x_0$$
$$y(t) = Cx(t) \tag{7}$$
$$x(t) \in R^n,\; u(t) \in R^r,\; y(t) \in R^m,\; m < n$$

The initial state x(0) is unknown. Functions $u(\cdot) \in [L^2[0,T]]^r$ and $y(\cdot) \in [L^2[0,T]]^m$.

Reconstruction of the initial condition x(0) of the state

We assume that control and output signals are measured on [0, T] interval, where T is fixed observation time. Our purpose is to determine the state x(0). The output of the system for t ϵ [0, T] is given by well-known formula:

$$y(t) = Ce^{At}\,x(0) + C\int_0^t e^{A(t-s)}Bu(s)\,ds \tag{8}$$

For calculation of unknown x(0) one should multiply both side of the equation (8) by $e^{A't}\,C'$

$$e^{A't}\,C'\,y(t) = e^{A't}\,C'\,C\,e^{At}\,x(0) + e^{A't}\,C'\,C\int_0^t e^{A(t-s)}\,B\,u(s)\,ds$$

Matrix $e^{A't}\,C'\,C\,e^{At}$ is square however, singular for any *t*. Let therefore integrate both side of this equation in [0, T] interval

$$\int_0^T e^{A't}\,C'\,y(t)\,dt = \int_0^T e^{A't}\,C'\,C\,e^{At}\,dt\;x(0) + \int_0^T\left[e^{A't}\,C'\,C\int_0^t e^{A(t-s)}\,B\,u(s)\,ds\right]dt$$

Real square and symmetric matrix M_0 is called the Gram Matrix.

$$M_0(T) = \int_0^T e^{A't}\,C'\,C\,e^{At}\,dt$$

This Gram matrix depends on observation time $M_0(T)$ but in sequel we will omit this notation and use M_0. If the system is observable (5), the real Gram matrix M_0 is non-singular for any T.

$$x(0) = M_0^{-1}\int_0^T e^{A't}\, C'\, y(t)\, dt - M_0^{-1}\int_0^T \left[e^{A't}\, C'\, C \int_0^t e^{A(t-s)}\, B\, u(s)\, ds \right] dt =$$

$$x(0) = \int_0^T M_0^{-1}\, e^{A't}\, C'\, y(t)\, dt - \int_0^T \left[\int_s^T M_0^{-1}\, e^{A't}\, C'\, C e^{A(t-s)}\, dt \right] B\, u(s)\, ds \qquad (9)$$

$$x(0) = \int_0^T \bar{G}_1(t)\, y(t)\, dt + \int_0^T \bar{G}_2(s)\, u(s)\, ds \qquad (10)$$

$$\bar{G}_1(t) = M_0^{-1}\, e^{A't}\, C' \qquad (11)$$

$$\bar{G}_2(s) = M_0^{-1} \left[\int_s^T e^{A't}\, C'\, C\, e^{At}\, dt \right] e^{-As}\, B \qquad (12)$$

The observer matrices $\bar{G}_1(T,t),\ \bar{G}_2(T,t)$ also depends on observation time T and on current time t but in sequel we will omit this notation and use $\bar{G}_1(t),\ \bar{G}_2(t)$.

Reconstruction of the final condition $x(T)$ of the state

In the similar way one can built the exact state observer of the final state $x(T)$, which is more useful for on-line control than $x(0)$. The output of the system (7) based on $x(T)$ has the form

$$y(t) = Ce^{-A(T-t)}x(T) - C\int_t^T e^{A(t-s)}Bu(s)ds \qquad (13)$$

For $x(T)$ calculation one should multiply both side of the equation (13) by the matrix $e^{-A'(T-t)}\, C'$ and integrate it on [0, T]. For observable system, the Gram matrix M_T is nonsingular for any T.

$$M_T = \int_0^T e^{-A'(T-t)}\, C'\, C\, e^{-A(T-t)}\, dt = e^{-A'T}\, M_0\, e^{-AT} \qquad (14)$$

$$x(T) = \int_0^T M_T^{-1}\, e^{-A'(T-t)}\, C'\, y(t)\, dt + \int_0^T \left[M_T^{-1} e^{-A'(T-t)}\, C'\, C \int_t^T e^{A(t-s)}\, B\, u(s)\, ds \right] dt$$

$$x(T) = \int_0^T M_T^{-1}\, e^{-A'(T-t)}\, C'\, y(t)\, dt + \int_0^T \left[M_T^{-1} \int_0^S e^{-A'(T-t)}\, C'\, C\, e^{A(t-s)}\, dt \right] B\, u(s)\, ds \qquad (15)$$

$$x(T) = \int_0^T G_1(t)\, y(t)\, dt + \int_0^T G_2(s)\, u(s)\, ds \qquad (16)$$

$$G_1(t) = M_T^{-1}\, e^{-A'(T-t)}\, C' = e^{AT}\, M_0^{-1}\, e^{A't}\, C' \qquad (17)$$

$$G_2(t) = \left[M_T^{-1} \int_0^t e^{-A'(T-\tau)}\, C'\, C\, e^{A(\tau-t)}\, d\tau \right] B = e^{AT} M_0^{-1} \left[\int_0^t e^{A'\tau}\, C'\, C\, e^{A\tau}\, d\tau \right] e^{-At}\, B \qquad (18)$$

The integral observer is given by two inner products in L^2[0, T] function spaces and after the first observation window the exact value of final state $x(T)$ is reconstructed. Multiplication of (13) by the above rectangular matrix $e^{A't}\, C'$, can be replaced by the multiplication of almost „any" other rectangular matrix, such that the obtained square matrix M will be non-singular. We see here an analogy to the instrumental variable method used in identification of discrete systems (which is a modification of the least squares method). Hence, it can be assumed that there is infinite number of the exact state observers for a given LTI process. In that case, one can look for the best observer (from some point of view i.e. under some quality index).

The above statement about the exactness in reconstruction of the state by the use of integral observer is valid of course only under the assumption of perfect input-output measurements i.e. without input-output disturbances and if the measurement noise is negligible. However, in measurement practice, the noisy measurements may occur and then the use of integral observers gives the reconstruction error.

$$\int_0^T G_1(t)\, [y(t) + z_1(t)]\, dt + \int_0^T G_2(s)\, [u(s) + z_2(t)]\, ds =$$

$$= \int_0^T G_1(t)\, y(t)dt + \int_0^T G_2(s)u(s)ds + \int_0^T G_1(t)\, z_1(t)dt + \int_0^T G_2(s)z_2(s)ds = x(T) + \varepsilon(T)$$

A reasonable selected observer will guarantee minimization of the state reconstruction error ε. It will be the observer with minimal norm in L^2[0, T]. To this end function matrices $G_1(t)$ and $G_2(t)$ should be carefully chosen from the class of all admissible observer matrices. The norms of presented above versions of the exact observers (11), (12) and (17), (18) are not minimal. Hence, the above versions of the exact state observers are not optimal in general case and are only special cases, which can by derived also from the general theory of the exact and optimal state observation. The general theory of optimal and exact state observation was formulated and presented by Byrski and Fuksa [7], [8].

Remarks on the structure of the observer:

The observer reconstructs the real unknown number (vector) $x \epsilon R^n$ based on two continuous pieces of functions u and y given on finite time interval [0, T]. Hence, the exact state observer must have the structure of two linear continuous functionals.

On the other hand from the Riesz Representation Theorem [9], it follows that every linear continuous functional in $L^2[0, T]$ space can be expressed as functions inner product given by the integral. Therefore, for SISO system the structure of the observer has to be given by two inner products on [0,T]: one product of continuous output function $y(\cdot) \epsilon Y$ and special observation vector function $G_1(\cdot) \epsilon Y^n$ (which must guarantee observability requirements), the second product of the input function $u(\cdot) \epsilon U$ and special observation vector function $G_2(\cdot) \epsilon U^n$.

In multidimensional case $y(\cdot) \in [L^2[0,T]]^m = Y$ and $u(\cdot) \in [L^2[0,T]]^r = U$ and so the functions G are matrices G_1(t), G_2(t). Function elements of these matrices have to have the shape, which will minimize the norm of the observer. This norm is defined in the space $Y^n x\ U^n$.

The norm of the observer

For the exact state observer

$$\begin{bmatrix} x_1(T) \\ \vdots \\ x_n(T) \end{bmatrix} = \int_0^T \begin{bmatrix} g_1^{11} & \cdots & g_1^{1m} \\ \vdots & \ddots & \vdots \\ g_1^{n1} & \cdots & g_1^{mn} \end{bmatrix} \begin{bmatrix} y_1(t) \\ \vdots \\ y_m(t) \end{bmatrix} dt + \int_0^T \begin{bmatrix} g_2^{11} & \cdots & g_2^{1r} \\ \vdots & \ddots & \vdots \\ g_2^{n1} & \cdots & g_2^{mr} \end{bmatrix} \begin{bmatrix} u_1(t) \\ \vdots \\ u_r(t) \end{bmatrix} dt \quad (19)$$

it will be assumed that the squared norm of the observer (19) will be given by (20).

$$J = \int_0^T \left[\sum_{i=1}^n \alpha_i \sum_{j=1}^m (g_1^{ij}(\tau))^2 + \sum_{i=1}^n \beta_i \sum_{j=1}^r (g_2^{ij}(\tau))^2 \right] d\tau \quad (20)$$

Interpretation 1: In this norm the weight factor α_i is connected with the norm of the matrix G_1 and represents the level of influence of the disturbances existing in all the output signals, to accuracy of reconstruction of i-th state variable $x_i(T)$**.** The weight factor β_i represents the level of influence of the all input disturbances, to accuracy of reconstruction of i-th state variable. For n=3 it is visible below in the equation:

$$J = \int_0^T \left[\begin{pmatrix} \alpha_1\left((g_1^{11})^2 + (g_1^{12})^2 + (g_1^{13})^2\right) + \\ + \alpha_2\left((g_1^{21})^2 + (g_1^{22})^2 + (g_1^{23})^2\right) + \\ + \alpha_3\left((g_1^{31})^2 + (g_1^{32})^2 + (g_1^{33})^2\right) \end{pmatrix} + \begin{pmatrix} \beta_1\left((g_2^{11})^2 + (g_2^{12})^2 + (g_2^{13})^2\right) + \\ + \beta_2\left((g_2^{21})^2 + (g_2^{22})^2 + (g_2^{23})^2\right) + \\ + \beta_3\left((g_2^{31})^2 + (g_2^{32})^2 + (g_2^{33})^2\right) \end{pmatrix} \right] dt$$

Interpretation 2: The optimal observer with minimal norm J guarantees minimal state reconstruction error, because this norm estimates the upper value of the possible error, if the disturbances in y and u are bounded and have the unit norm (belong to the unit balls) and are the "worst" type disturbances. The norm (20) may be treated as the quality index of the observation

Below dependencies between the norm and the observation error will be presented.

$$\varepsilon(T) = \int_0^T G_1(\tau) z_1(\tau)\, d\tau + \int_0^T G_2(\tau)\, z_2(\tau)\, d\tau \quad (21)$$

$$\max_{(z_1,z_2)} \|\varepsilon\|_{R^n}^2 = \max_{(z_1,z_2)} \left\| \int_0^T G_1(\tau) z_1(\tau)\, d\tau + \int_0^T G_2(\tau)\, z_2(\tau)\, d\tau \right\|^2 \le$$

$$\le 2\left[\max_{\|z_1\|=1} \left\| \int_0^T G_1(\tau) z_1(\tau)\, d\tau \right\|^2 + \max_{\|z_2\|=1} \left\| \int_0^T G_2(\tau)\, z_2(\tau)\, d\tau \right\|^2 \right] \stackrel{def}{=}$$

$$\stackrel{def}{=} 2\left[\|G_1\|_{Y^n}^2 + \|G_2\|_{U^n}^2 \right] = 2\|(G_1, G_2)\|_{Y^n \times U^n}^2 = 2\, J$$

$$\min_{(G_1,G_2)} \left[\max_{(z_1,z_2)} \|\varepsilon\|_{R^n}^2 \right] = \min_{(G_1,G_2)} J = J(G_1^o, G_2^o)$$

Hence, the assumed form of the squared norm of the observer can represents performance index of observation, which should be minimized. The optimization task is:

$$\min_{(G_1,G_2)} J = \min_{(G_1,G_2)} \int_0^T \left[\sum_{i=1}^n \alpha_i \sum_{j=1}^m (g_1^{ij}(\tau))^2 + \sum_{i=1}^n \beta_i \sum_{j=1}^r (g_2^{ij}(\tau))^2 \right] d\tau \quad (22)$$

The general conditions for the existence of the exact state observer

The existence conditions will be derived for final state observer. The output formula (13)

$$y(t) = Ce^{-A(T-t)} x(T) - C\int_t^T e^{A(t-s)} Bu(s) ds\,,$$

will be substitute to the observer formula (16)

$$x(T) = \int_0^T G_1(\tau)\; y(\tau)\, d\tau + \int_0^T G_2(\tau)\; u(\tau)\, d\tau\,.$$

Then one can obtain

$$x(T) = \int_0^T G_1(\tau) \left[C\, e^{-A(T-\tau)}\, x(T) - C\int_\tau^T e^{A(\tau-s)}\, B\, u(s)\, ds \right] d\tau + \int_0^T G_2(\tau)\, u(\tau)\, d\tau\,,$$

$$x(T) = \int_0^T G_1(\tau)\, C\, e^{-A(T-\tau)}\, d\tau \cdot x(T) - \int_0^T G_1(\tau) \left[C\int_\tau^T e^{A(\tau-s)}\, B\, u(s)\, ds \right] d\tau + \int_0^T G_2(\tau)\, u(\tau)\, d\tau\,.$$

After changing the order of two integrations, the formula (23) is obtained.

$$x(T) = \int_0^T G_1(\tau) C e^{-A(T-\tau)}\, d\tau \cdot x(T) - \int_0^T \left[\int_0^s G_1(\tau) C e^{A(\tau-s)} B\, d\tau \right] u(s)\, ds + \int_0^T G_2(\tau) u(\tau)\, d\tau \quad (23)$$

It is easy to see that the general conditions for (23) to be an exact state observer are:

The G_1 matrix should fulfil the condition

$$\int_0^T G_1(\tau) C e^{-A(T-\tau)} d\tau = I \quad (24)$$

The G_2 matrix should fulfil the condition

$$G_2(\tau) = \int_0^\tau G_1(s)\; C\; e^{-A(\tau-s)}\, B\; ds \quad (25)$$

The formula (24) represents the mathematical constraint for the all-possible matrices G_1 that can be observation matrices. If such a matrix G_1 will be chosen, then the matrix G_2 must fulfil the equation (25). The observer (16) with such a matrices G_1 and G_2 will have the norm (20).

Minimization of the norm (22) (with assumed factors $\alpha_i > 0$, $\beta_i \ge 0$) on the set of all admissible matrices G_1 and G_2 gives the optimal exact state observer with minimal norm. The matrices of this observer of course, must fulfil conditions (24) and (25).

The general formulas for the optimal exact state observer for any assumed $\alpha_i > 0$, $\beta_i \ge 0$, have been first time presented in 1984, in the publications [7], [8].

Two special cases for the exact final state observers

The first special case for the factors $\alpha_i = 1$, $\beta_i = 0$, may be considered for the situation when the minimization of the matrix G_2 norm is not needed. This is when the control signal is known (there is no need of its measurement), as well as there is no disturbances z_2 in this signal, which could cause an extra state estimation error (21). Hence, the norm (20) has the simplified form

$$\|(G_1,G_2)\|^2 = \int_0^T \left[\sum_{i=1}^{n}\sum_{j=1}^{m} (g_1^{ij}(\tau))^2 \right] d\tau = J \tag{26}$$

$$J^o = \min_{(G_1)} J$$

After substitution of $\alpha_i = 1$, $\beta_i = 0$, to general optimal observer formula [7], on can obtain:

$$G_1^o(t) = M_T^{-1}\, e^{-A'(T-t)}\, C' = e^{AT}\, M_0^{-1}\, e^{A't}\, C'$$

where $M_T = e^{-A'T}\int_0^T e^{A't}\, C'C\, e^{At}\, dt\, e^{-AT}$

$$G_2^o(t) = \left[M_T^{-1} \int_0^t e^{-A'(T-\tau)}\, C'\, C\, e^{A(\tau-t)}\, d\tau \right] B = e^{AT} M_0^{-1} \left[\int_0^t e^{A'\tau}\, C'\, C\, e^{A\tau}\, d\tau \right] e^{-At}\, B \tag{27}$$

It turned out that these are the same formula like in (16), (17).

The second special case is for $\alpha_i = 1$, $\beta_i = 1$, and may be useful in situation when in both signals, control u and the output y, the similar measurement disturbances may occur and both can affect the value of state estimation error. Hence, the full norm of the observer should be minimized.

$$\|(G_1,G_2)\|^2 = \int_0^T \left[\sum_{i=1}^{n}\sum_{j=1}^{m} (g_1^{ij}(\tau))^2 + \sum_{i=1}^{n}\sum_{j=1}^{r} (g_2^{ij}(\tau))^2 \right] d\tau = J \tag{28}$$

$$J^o = \min_{(G_1,G_2)} J$$

After substitution of $\alpha_i = 1$, $\beta_i = 1$, to general optimal observer formula [7], on can obtain:

$$\begin{aligned} G_1^o(\tau) &= M_T^{-1}\, \Phi_{11}'(\tau)\, C' \\ G_2^o(\tau) &= M_T^{-1}\, \Phi_{21}'(\tau)\, B. \end{aligned} \tag{29}$$

$$M_T^{-1} = e^{AT}\left[\int_0^T \Phi_{11}'(\tau)\, C'\, C\, e^{A\tau}\, d\tau \right],$$

where matrices $\Phi_{11}(t)$ and $\Phi_{21}(t)$ are calculated as submatrices of fundamental matrix *exp(Wt)*.

$$W = \begin{bmatrix} A & BB' \\ C'C & -A' \end{bmatrix}, \qquad \Phi(\tau) = e^{W\tau} = \begin{bmatrix} \Phi_{11}(\tau) & \Phi_{12}(\tau) \\ \Phi_{21}(\tau) & \Phi_{22}(\tau) \end{bmatrix}$$

$$x(T) = \int_0^T \left[M_T^{-1}\, \Phi_{11}'(\tau)\, C' \right] y(\tau)\, d\tau + \int_0^T \left[M_T^{-1}\, \Phi_{21}'(\tau)\, B \right] u(\tau)\, d\tau \tag{30}$$

Application of the exact state observer in on-line mode

The main formula for the final state observer (16)

$$x(T) = \int_0^T G_1(\tau)\, y(\tau)\, d\tau + \int_0^T G_2(\tau)\, u(\tau)\, d\tau$$

is valid for every T and for any output y and input u, so also for arbitrarily shifted y and u.

$$x(t) = \int_0^T G_1(\tau)\, y(t-T+\tau)\, d\tau + \int_0^T G_2(\tau)\, u(t-T+\tau)\, d\tau$$

After changing limits of integration (Figure 1) it creates the Moving Window Observer – MWO.

$$x(t) = \int_{t-T}^{t} G_1(T-t+\tau)\, y(\tau)\, d\tau + \int_{t-T}^{t} G_2(T-t+\tau)\, u(\tau)\, d\tau \tag{31}$$

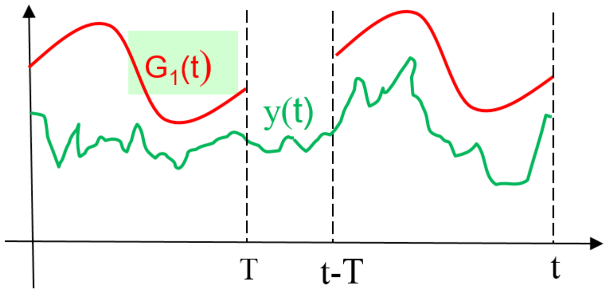


Figure 1. Moving Window Observer

One can use the optimal integral observer in on-line state reconstruction as a moving observation window observer with fixed width T of the windows, shifted along time axis. After multiplication of functions $G_1(\tau)$ by $y(\tau)$ and $G_2(\tau)$ by $u(\tau)$ measurements on $[t-T, t]$ and after integration of these products on $[t-T, t]$, it gives the exact value of the state $x(t)$ for $\forall t \geq T$, [10]. Hence, one can see that in on-line mode (for the current time *t*), the application of the exact state observers need devices with more computation power than for the application of Kalman Filter estimator. It is also visible, that the exact state reconstruction is possible only for $t \geq T$.

The important problem for the integral observer is the choice of window width T. For different T we have different $G_1(t)$, $G_2(t)$, which can be calculated off-line. If there are no disturbances in measurements, the exact state reconstructed value does not depend on the interval T and generally on the norm of the observer. Hence, from computation effort point of view the window width T should be as small as possible (it gives also the reduction of the delay in the exact reconstruction x(T) after the first window). On the other hand, the main statement is that the norm of the observer depends on T. It was turn out that with decreasing T, the observer's norm increases and when T tends to zero the norm of observer tends to infinity. In disturbed measurements case, the observer should have the smaller norm and hence be calculated for the bigger interval T. One can determine even the minimum value of T to guarantee admissible state reconstruction error. For different levels of disturbances, one can prepare also, the set of the different observers for different values of observation times T_i and use adaptive switching.

Analytical example for the integral MWO

Let the LTI second order SISO system be given, as in previous examples.

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x(t) + Bu(t) \qquad (32)$$

$$y(t) = \begin{bmatrix} 2 & 0 \end{bmatrix} x(t)$$

The integral state observer has the form of two matrices (in this case vectors) G_1 and G_2.

$$\begin{bmatrix} x_1(T) \\ x_2(T) \end{bmatrix} = \int_0^T \begin{bmatrix} G_{11}^o(\tau) \\ G_{12}^o(\tau) \end{bmatrix} y(\tau)\, d\tau + \int_0^T \begin{bmatrix} G_{21}^o(\tau) \\ G_{22}^o(\tau) \end{bmatrix} u(\tau)\, d\tau \qquad (33)$$

For this system based on formula (16), (17) or (27) the optimal observer matrices for the first case (α=*1,* β=*0*) are given by (34).

$$G_1^o(t) = \frac{1}{T^3} \begin{bmatrix} 3T\,t - T^2 \\ 6t - 3T \end{bmatrix}, \quad G_2^o(t) = \frac{1}{T^3} \begin{bmatrix} T^2\,t^2 - T\,t^3 \\ 3T\,t^2 - 2\,t^3 \end{bmatrix} \qquad (34)$$

$$\|(G_1, G_2)\|(T) = \sqrt{\int_0^T \left[[G_{11}(t)]^2 + [G_{12}(t)]^2 \right] dt} = \sqrt{\frac{1}{T} + \frac{3}{T^3}}, \qquad (35)$$

This norm as the function of T is visible on the Figure 2.

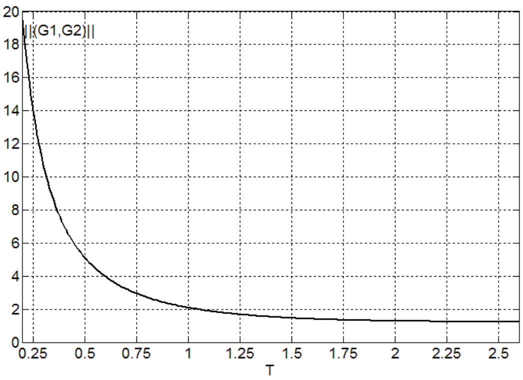


Figure 2. The norm (35) of the observer as the function of T

Based on formula (29) one can find the optimal observer matrices for the second case *(*α=*1,* β=*1).*

$$W = \begin{bmatrix} A & BB' \\ C'C & -A' \end{bmatrix}, \qquad \Phi(t) = e^{W\tau} = \begin{bmatrix} \Phi_{11}(\tau) & \Phi_{12}(\tau) \\ \Phi_{21}(\tau) & \Phi_{22}(\tau) \end{bmatrix}$$

$$\Phi_{11}(t) = \begin{bmatrix} \cosh t \cdot \cos t, & (\sinh t \cos t + \cosh t \cdot \sin t)/2 \\ \sinh t \cos t - \cosh t \cdot \sin t, & \cosh t \cdot \cos t \end{bmatrix}$$

$$\Phi_{21}(t) = \begin{bmatrix} 2(\sinh t \cdot \cos t + \cosh t \cdot \sin t), & 2 \cdot \sinh t \cdot \sin t \\ -2 \cdot \sinh t \cdot \sin t, & \sinh t \cdot \cos t - \cosh t \cdot \sin t \end{bmatrix}$$

The final formula for the observer optimal matrices (29)

$$G_1^o(t) = M_T^{-1} \begin{bmatrix} 2\cosh t \cdot \cos t \\ \sinh t \cdot \cos t + \cosh t \cdot \sin t \end{bmatrix}$$
$$G_2^o(t) = M_T^{-1} \begin{bmatrix} -2 \cdot \sinh t \cdot \sin t \\ \sinh t \cdot \cos t - \cosh t \cdot \sin t \end{bmatrix} \qquad (36)$$

where the inversion of Gram matrix is equal to

$$M_T^{-1} = \frac{1}{2\sinh^2 T - 2\sin^2 T} \begin{bmatrix} \sinh T \cdot \cos T - \cosh T \cdot \sin T, & 2 \cdot \sinh T \cdot \sin T \\ -2 \cdot \sinh T \cdot \sin T, & 2 \cdot (\sinh T \cdot \cos T + \cosh T \cdot \sin T) \end{bmatrix}$$

and the observer's norm is given by (37)

$$\|(G_1^o, G_1^o)\| = \sqrt{\frac{3\sinh 2T + \sin 2T}{4(\sinh^2 T - \sin^2 T)}}, \quad \|(G_1, G_2)\|(\infty) = \sqrt{1.5}. \qquad (37)$$

In Figure 3, one can see the dependence of the norm of estimation error on time observation T. This error is estimated by the norm of the observer $\|\varepsilon\|(T) \le \sqrt{2} \cdot \|(G_1, G_2)\|(T)$. The error is presented for two different state observers (34) and (36). The lowest graph is for the simplified case 1: $\alpha_i = 1$, $\beta_i = 0$, given by the formula (27), (34). A bit higher located graph represents the full norm for the case 2: $\alpha_i = 1$, $\beta_i = 1$, formula (37). And the highest located graph represents the estimation of the worst error for the case 1, when despite of the expected noise absence in the control signal, such a worst disturbances from the unite ball in u(t) will appear. However, in this case the matrix $G_2(t)$ is not optimal and so contributes to significant errors. This unexpected error is presented by third graph and the formula

$$\|\varepsilon\| \le \sqrt{2} \cdot \|(G_1, G_2)\| = \sqrt{2} \cdot \sqrt{\int_0^T \left[[G_{11}(t)]^2 + [G_{12}(t)]^2 + [G_{21}(t)]^2 + [G_{22}(t)]^2 \right] dt} =$$

$$= \sqrt{2} \cdot \sqrt{(T^6 + 39T^4 + 105T^2 + 315)/(105\,T^3)} \qquad (38)$$

It seems that for the system (32) and the guaranteed case 1 ($\alpha_i = 1$, $\beta_i = 0$) the use of the observer (34) with the best observation time T϶[4 – 6] gives and the smallest error. For the case 1 and possibility of control signal noise occurrence the best observation time is T϶[2 – 3]. For the case 2 ($\alpha_i = 1$, $\beta_i = 1$) the most reasonable is the use of the observer (36) with observation time T϶[2 – 3].

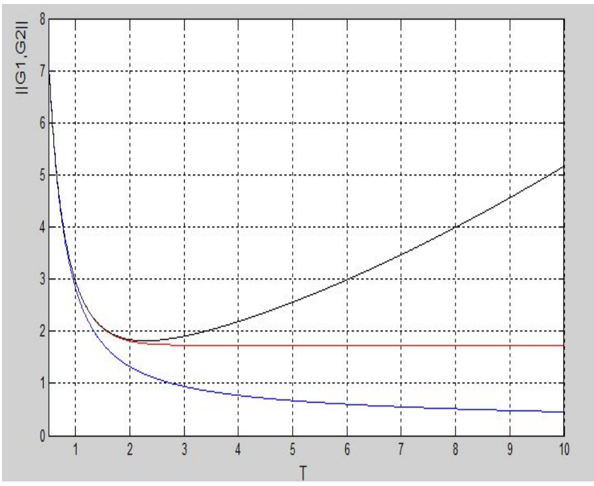


Figure 3. The estimation error given by the different observers as the function of T.
Blue curve (lowest) for case 1 (35), red curve (higher) for case 2 (37), black curve (highest) for the worst error estimation in case 1 (38)

In the Figure 4. the shape of the optimal observer matrices G_1 and G_2 (29) for T=2 are visible. In the Figures 5, 6, 7, 8, 9, 10 one can see the application of the observer (29) and the superiority of the MWO (31) over the asymptotic observers in on-line observation of the system (32) as well as in its stabilization by the use of Linear Quadratic Regulator (LQR).

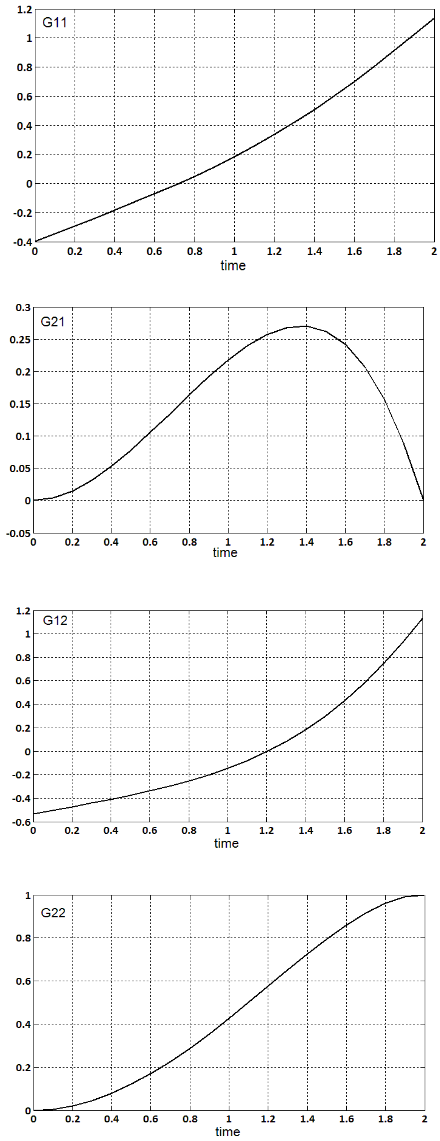
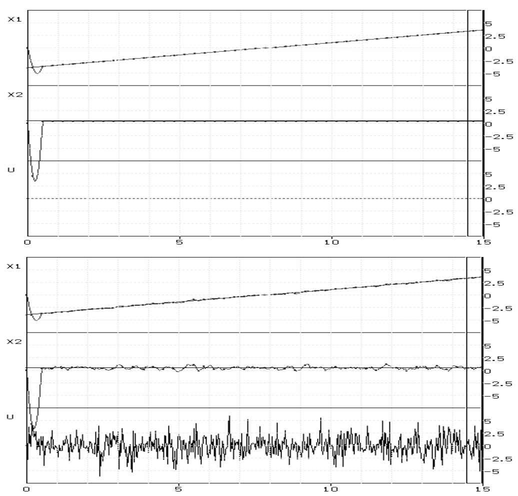


Figure 4. The optimal observer matrices G_1 and G_2 for T=2.0 from (29)



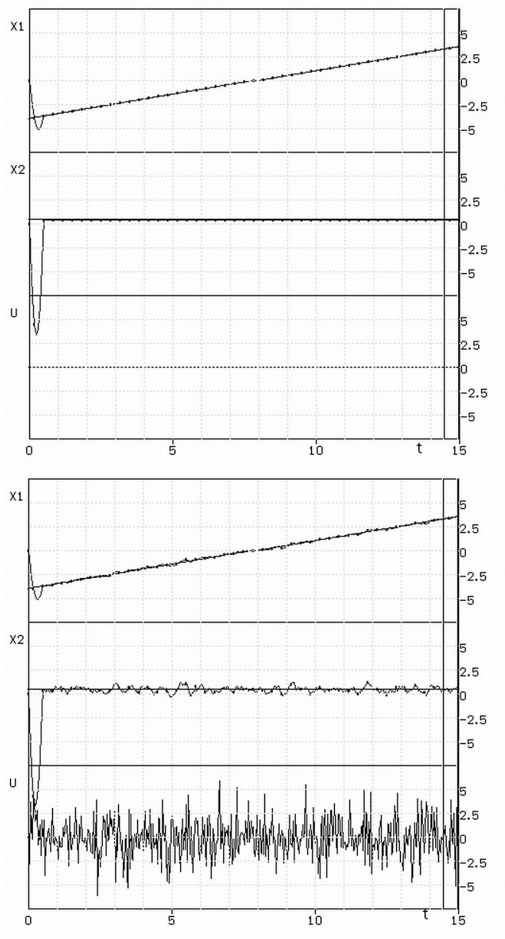


Figure 5. Reconstruction of the state variables x_1, x_2 without measurement noise in u(t)=1(t) (upper plots) and with measurement noise (lower plots). Left Fig. for MWO T=0.5, right Fig. for MWO, T=1.0

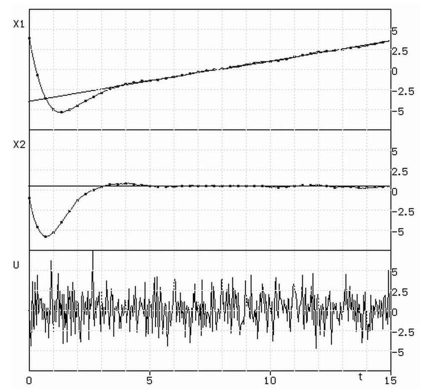


Figure 6. Estimation of the state variables (32) with the measurement noise in control by the use of Kalman Filter

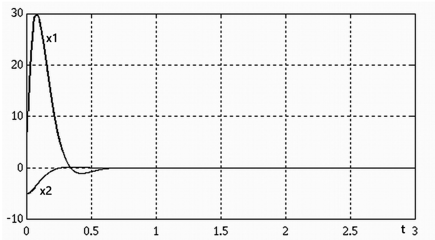


Figure 7. Stabilization of the state in system (32) from nonzero initial conditions with the use of LQR and direct measurement of the state x1(t) and x2(t) (without the observer)

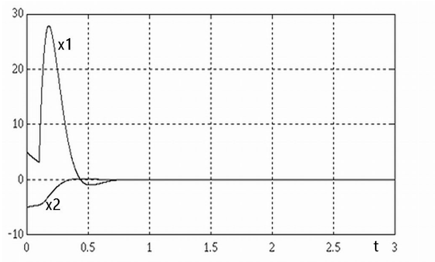


Figure 8. Stabilization of system (32) with LQR and reconstructed state x1(t) and x2(t) by the use of MWO observer T=0.1

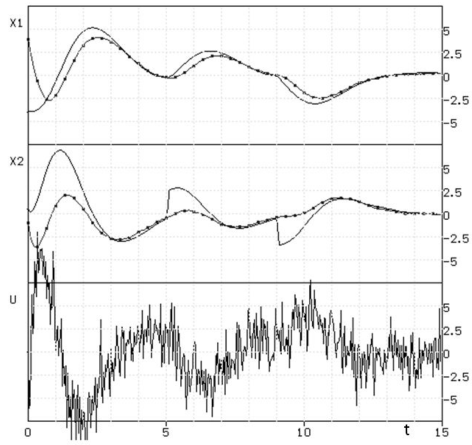


Figure 9. State stabilization with LQR and KF MWO with measurement noise

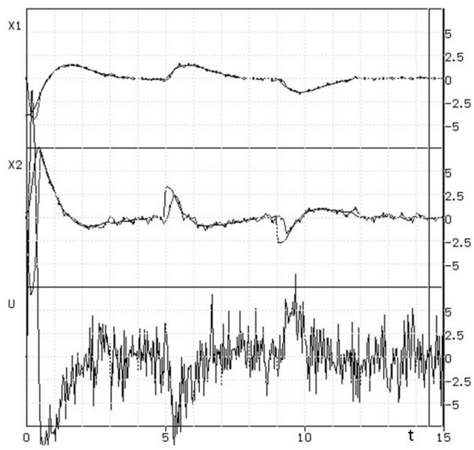


Figure 10. State stabilization with z LQR and MWO with T=0.5, with measurement noise

Discrete version of the exact state observers

Now will be derived the formulas for the discrete version of the observer for the exact state reconstruction. The discretization time sample is Δ. Let us assume the SISO system and the second case of the exact state observer it means with coefficients $\alpha=\beta=I$.

The vector samples of y(t) and u(t) consist of N+1 measurements on interval [0,T]. Hence, the output, input and the state vector spaces will be $Y=R^{N+1}$, $U=R^{N+1}$, $X=R^{n}$, respectively.

The final state x(N) is unknown and should be reconstructed. The system equation is given by:

$$y = H_1\, x(N\Delta) + H_2\, u$$

$$\begin{bmatrix} y_0 \\ \vdots \\ y_{N\Delta} \end{bmatrix} = H_1 \begin{bmatrix} x_1(N\Delta) \\ \vdots \\ x_n(N\Delta) \end{bmatrix} + H_2 \begin{bmatrix} u_o \\ \vdots \\ u_{N\Delta} \end{bmatrix} \tag{39}$$

In i-th sample of discretization $i\Delta$ the equation (39) has the form

$$y(i\Delta) = Ce^{-A(N-i)\Delta}x(N\Delta) - Ce^{Ai\Delta}\int_{i\Delta}^{N\Delta} e^{-A\tau}Bu(\tau)d\tau$$

$$\begin{bmatrix} y_0 \\ . \\ y_{i\Delta} \\ . \\ y_{N\Delta} \end{bmatrix} = \begin{bmatrix} Ce^{-AN\Delta} \\ . \\ Ce^{-A(N-i)\Delta} \\ . \\ C \end{bmatrix} x(N\Delta) + H_2 \begin{bmatrix} u_0 \\ . \\ . \\ . \\ u_{N\Delta} \end{bmatrix} \tag{40}$$

The forms of matrices H_1 and H_2 follow from the above equation.

The form of the matrix H_2 one can simplified by the use of the new variable $s=\tau-j\Delta$.

$$H_2(i\Delta)u = -Ce^{Ai\Delta}\left(\sum_{j=i}^{N-1}\int_{j\Delta}^{(j+1)\Delta} e^{-A\tau}Bu(\tau)d\tau\right) = -Ce^{Ai\Delta}\left(\sum_{j=i}^{N-1} e^{-Aj\Delta}\int_{0}^{\Delta} e^{-As}Bu(s+j\Delta)\,ds\right)$$

Assuming existence of the zero order hold ZOH (conventional digital-to-analog converter - DAC) which holds the value of each control signal sample for one sample interval $u(j\Delta)$=const, one can calculate the integral in H_2 and finally obtain

$$H_2 = \begin{bmatrix} -CB_D & -Ce^{-A\Delta}B_D & . & . & -Ce^{-A(N-1)\Delta}B_D \\ 0 & -CB_D & . & . & -Ce^{-A(N-2)\Delta}B_D \\ . & . & . & . & . \\ . & . & . & . & . \\ 0 & . & . & . & -CB_D \end{bmatrix}_{N\times N}, \quad B_D = \int_0^{\Delta} e^{-As}Bds \tag{41}$$

The discrete state observer is given by two matrices $G_1[n\times(N+1)]$, $G_2[n\times(N+1)]$:

$$x(N\Delta) = G_1 y + G_2 u \tag{42}$$

The norm of the observer is given by the sum of the inner products:

$$\left\|\left(G_1^T, G_2^T\right)\right\| = \frac{1}{\Delta}\sqrt{\left[\sum_{i=1}^{n}\left\langle g_1^i, g_1^i\right\rangle_Y + \sum_{i=1}^{n}\left\langle g_2^i, g_2^i\right\rangle_U\right]} \quad (43)$$

The forms of the optimal observer matrices, which minimizes the norm (43) one can numerically find from the below formulas (44), [7], [8], [17]:

$$G_1^o = \left(H_1^T F^{-1} H_1\right)^{-1} H_1^T F^{-1} \quad , \quad G_2^o = -G_1^o H_2 \quad (44)$$

where $F = \left(I + H_2 H_2^T\right)$.

The vector of unknown final state x(NΔ) in given interval one can find from the formula.

$$\begin{bmatrix} x_1(N\Delta) \\ \vdots \\ x_n(N\Delta) \end{bmatrix} = G_1 \begin{bmatrix} y_0 \\ \vdots \\ y_{N\Delta} \end{bmatrix} + G_2 \begin{bmatrix} u_o \\ \vdots \\ u_{N\Delta} \end{bmatrix} \quad (45)$$

These matrices G_1 and G_2 can be also use in on-line moving window discrete observer.

Results and Discussion

All the above discussion on the integral exact state observers proved their superiority over the asymptotic estimators. Three useful remarks about the application of the adaptive exact state observers may be done.

1. Taking into account two facts that the basic observation problem is the reconstruction of the vector state with minimal error as well as that the noises are always acting in measurements, undoubtedly, the observer with a minimal norm should be designed. The problem strictly related to this, is the proper choice of the observation time T. If the norm of disturbances are small and the noise may be neglected, observation time T may be short because the norm of the observer does not matter. If the norm of disturbances is significant, the time T should be long enough that the observer norm would be not to big. If the norm of disturbances varies in time, the adaptive version of the on-line observers may be proposed. Current identification of the disturbances and their norm enables calculation of minimum window's width T_{min}, which will guarantee admissible reconstruction error. The method of disturbance identification was given in [11].
2. There are also possible the other adaptive exact state reconstruction strategies. In advance (off-line) one can prepare a few observer's pairs (G_{1i}, G_{2i}) (the bank of observers with different norms) for different windows width Ti <Ti+1 and start the state reconstruction process with the use of the shortest window. If however, the norm of the noise is significant, then for state observation the observers should be consecutively exchanged, and have longer and longer windows (the observer switching structure).
3. The third adaptive strategy is based on the cooperation of different observers. If the controlled object is very fast (e.g. electric drive control) and computation power of used control devices is not enough, the integral MWO can cooperate with KF algorithm [12]. For current state estimation, the Kalman Filter can be used and the exact state observer with very short window runs simultaneously in parallel. After the first window T_1, when MWO reconstructs the exact state $x(T_1)$, it provides this value $\bar{x}(T_1) = x(T_1)$ to KF. Then the KF estimation is restarted. The LQR stabilizing regulator works all the time based on KF state estimation. The MWO exact observer is switched-on only occasionally when the LQR regulator will detect large increase in the control error (due to possible disturbances) and hence, the state estimate correction probably will be needed.

Conclusions

In this paper, the theory of the optimal asymptotic state estimation and the exact reconstruction of the state vector in finite time interval was recalled and compared. Numerical examples proved that the state estimation and the state stabilization by the use of exact observers are much efficient than the use of asymptotic estimators.

The asymptotic behavior of the convergence of the state estimate given by KF is due to its structure based on linear differential equation. The power of modern computers makes the application of the other on-line optimal observation algorithms possible. They originate directly from the definition of the exact observability and guarantee the exact state reconstruction of LTI system in finite time. The general theory of the deterministic approach to optimal exact state observation, for which the relations were formulated in Hilbert function spaces, was in [7], [8]. The structure of the observer is given by two inner products (integrals) of the output and input measurements and special observation functions $G_1(t)$, $G_2(t)$ on interval [0,T]. The optimal functions G(t) were chosen in such a way that they minimized the norm of the observer. The observer with minimal norm guarantees the minimal state reconstruction error for the worst disturbances taken from the unit balls in the measurements of both $y \in Y$ and $u \in U$. The state observer can be used in on-line mode as MWO. The choice of the width T of the observation window is an important problem for the integral observer. If there are no disturbances in measurements, the exactness of state reconstruction does not depend on the norm of the observer and so generally on T. Hence, from computation effort point of view and for decreasing the start delay of the exact reconstruction, the width T should be as small as possible. On the other hand, if in the measurements the disturbances will occur, the estimation error will depend on the norm of the observer. The main statement is that the norm of the observer depends on T. The norm increases with decreasing T. When T tends to zero the norm tends to infinity. Hence, in practice for disturbed measurements of y and u the reconstruction error will be less for bigger T. Therefore, the minimum value of T, which will guarantee of admissible state reconstruction error, can be calculated. In work [13] one can find an extra generalization of the state exact observation and the new formula for optimal observation matrices G_1, G_2 for the case when measurement disturbances exist and belongs to ellipsoids (and not to unit balls). In this work also, the differential version of the exact state observer, equivalent to integral MWO window state observer was derived, however it is a differential equation with lumped and distributed delay. The authors have elaborated also the other versions of the exact state observers and their applications and published the results of their research. One can find e.g. in [14] – problem of stabilization of the state in distillation column, in [15] – the state observation in distributed parameter system given by the heat equation, in [16] and [17] – the discrete version of the exact observers, and in [1] and [18] – the exact state observer working in the structure of double window for the fault detection.

The idea of using the exact state observers for various control tasks has also been noticed by other authors. Quoting various works chronologically, one can indicate: the application to nonlinear systems [19], to systems with delay [20], for the fault detection [21], [22], [23], to exact observation by differential observer [24], for time-derivative estimation [25], and others [26], [27].

Assuming, the most important properties of the integral state observers are:

- exact state observation for continuous linear systems,
- deterministic statement of the optimal observation problem in L^2 spaces,
- integral description of the on-line observer,
- high filtering properties of measuring noise due to the integral structure of the observer,
- fixed finite observation time interval T (of any length),
- independence on the state initial conditions x(0),
- off-line calculation of optimal matrices G_1 and G_2,
- on-line application as MWO,
- optimality of the observer from the point of view of noisy measurements in both u(t) and y(t) signals,
- possibility of calculation the minimal norm observer,
- the closed-loop system with integral observer and static LQ controller has the same eigenvalues as the system with direct measurement of the state and LQ controller only i.e. that the order n of the closed system with the MWO + LQR is the same as without the observer. The closed-loop feedback system with Kalman Filter has the order equal to 2n [7].
- The theory of the exact state observers can be formulated in Hilbert spaces [16] and applied to any linear system like system with distributed parameters or time delay if only the finite dimension state vector or any parameter $x \in R^n$ should be reconstructed.

Acknowledgments

This work was supported by the scientific research funds from the Polish Ministry of Science and Higher Education within the AGH UST Agreements no 16.16.120.773 and was also conducted within the research of EC Grant H2020-MS-CARISE-2018/824046.

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