

# State estimators and observers for continuous and discrete linear systems. Part 1. Differential asymptotic state estimators

Jędrzej Byrski<sup>a</sup>, Witold Byrski<sup>b,\*</sup>

<sup>a</sup>AGH University of Science and Technology, Department of Applied Computer Science, Al. Mickiewicza 30, Krakow 30-305, Poland

<sup>b</sup>AGH University of Science and Technology, Department of Automatic Control and Robotics, Al. Mickiewicza 30, Krakow 30-305, Poland

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## Abstract

In the paper an overview of state estimators and state observers used in linear systems, will be presented. The state estimators and observers can be used in many applications like the state reconstruction for the control purposes or for the diagnosis and fault detection in technical processes or for the virtual measurements of inaccessible variables of the system as well as for the best filtration of the differential equation solution. As the standard most commonly the Kalman filter and Luenberger type observers are used. Although the Kalman filter guarantees optimal filtering quality of the state, reconstructed from the noisy measurements, both Kalman filter and the Luenberger observer guarantee only asymptotic quality of the real state changes and tracking, basing on the current measurements of the system output and input signals. Unfortunately, the value of the estimation error at any moment of time cannot be calculated. The discussion on differences between continuous and two types of discrete Kalman Filter will be presented. This paper is planned to be the introduction to presentation of another type of the state observers which have the structure given by the integral operators. Based on measurements of the system output and input signals on some predefined finite time interval, they can reconstruct, after this interval, the observed state exactly.

**Keywords:** Linear estimators, Kalman filter, Luenberger observer, exact state observers, linear systems

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## Introduction

Stabilization of selected process variables by regulators is the basic task of the control system. Such variables are called the output vector of the system and in most cases, they are measured by sensors and transducers and are fed to output regulators (Programmable Logic Controllers PLC) working in feedback that produce a control signals. However, because of the high order of the system in the input-output path as well as the process disturbances, this task is not always as perfect as technologist would expect. In common approach, the real physical systems are approximated by the linear time invariant models (LTI) with lumped parameters. The high order of such a model is a consequence of the presence of  $n$  independent storage elements (energy, mass or momentum) in the structure of the system and depends on the choice of the physical variable, which is to be the output variable. Therefore, perfect stabilization of the level in the last tank placed in a series structure of four tanks while the control signal is the flow of the liquid to the first tank, is not an easy task Figure 1.

It can be done more accurately when the computer receives current information about the all tank levels (the state variables)

instead of the output, only, Figure 2. Then the control of the remaining levels is also possible and thus the output can be stabilize more precisely and quickly.

Unfortunately, such full instrumentation is not always possible in industrial systems. Hence, the important question arise – is it possible to stabilize the level of any chosen tank, based on the measurement of the last one, only. The answer is positive if the system fulfil the observability condition and the special algorithm (device) for state reconstruction is used. This algorithm is called the state observer and its structure is based on the differential model of the system, Figure 3.

In 1959 Rudolf Kalman (1930–2016) presented the theory of the optimal state filtration, for linear time invariant systems starting with zero state initial conditions. The observed system works in presence of input/output signal measurement noises of known covariance matrices. It enables designing of the filter differential equation with special optimal gain matrix. The filter estimates the vector state and minimizes the variance of the state estimation error (Linear Quadratic Filter, LQF). The first two publications on optimal state filtration were concerned to discrete systems [1, 2]. Two years later, Rudolf Kalman with Richard Bucy, have published this theory for the continuous system [3]. For the formulation of the filtration task and for the filter optimal solution finding, the stochastic processes as the

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\*Corresponding author: wby@agh.edu.pl

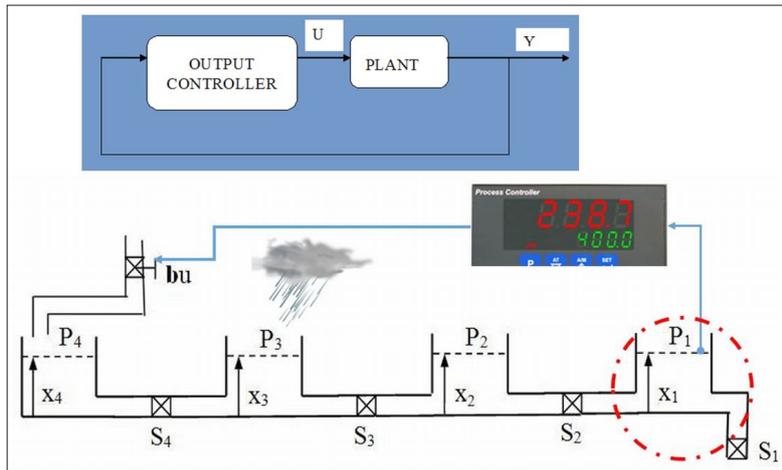


Figure 1. Stabilization of the measured output

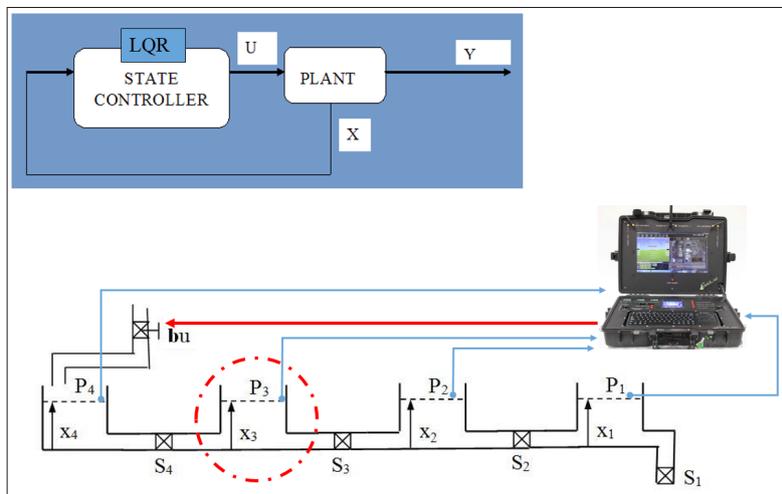


Figure 2. Stabilization of the all tank levels by the state controller

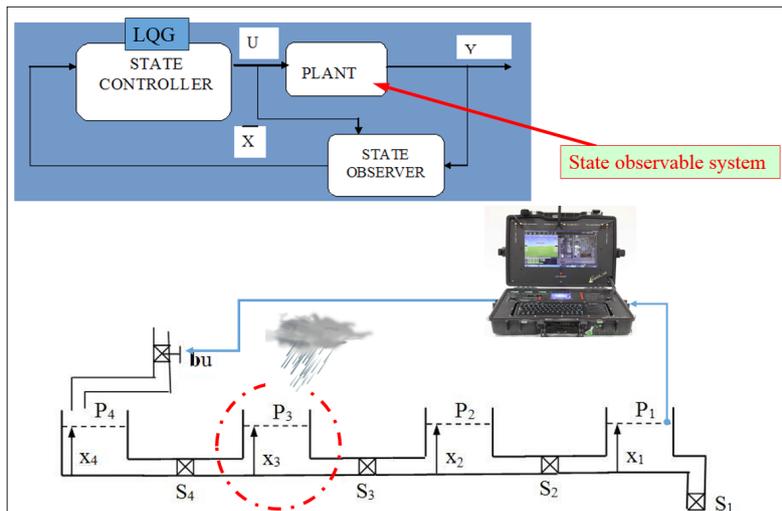


Figure 3. Stabilization of any tank level by the output controller and the state observer

mathematical background were used. Since 1970s, this type of filtration is called a Kalman Filter method (KF).

In 1961 William Linvill (Stanford University) suggested to the PhD student David Luenberger, the use of Kalman filter dif-

ferential equation for the asymptotic estimation of the system state, working without any stochastic noises, which has been started however, with unknown and nonzero initial conditions. Observing input and output signals of the system such type of

the deterministic state observer could estimate the current state vector with asymptotic convergence of the estimate to the real state of the system, however, with an unknown rate. In 1963, Luenberger defended in Stanford University a doctoral dissertation “*Determining the State of Linear System with Observers of Low Dynamic Order*”. The research results were published in [4]. Over the years, the simpler theory of Luenberger’s deterministic observer (LO) in full and reduced order versions has become more popular in automation and state observation applications, than the more advanced Kalman filtration theory. Sometimes even in some textbooks where Luenberger observer theory is described there is no reference to Kalman filter theory as the source idea. The main disadvantage of the use KF or LO for state estimation is the asymptotic convergence of the state estimate and lack of the knowledge about the estimation error value. In the next paper (Part 2) we will present quite different on-line observation algorithms, which can reconstruct the exact value of the state vector, e.g.  $x(t_0)$  at the moment  $t_0$ , making calculation on time interval  $[t_0, t_0+T]$ .

### State observability condition in linear time invariant systems

Let the linear model of homogeneous system be given,

$$\begin{aligned} \dot{x}(t) &= A x(t), & x(0) &= x_0 \\ y(t) &= C x(t) \end{aligned} \tag{1}$$

Where  $x(t) \in R^n$ ,  $u(t) \in R^r$  and  $y(t) \in R^m$ , for  $\forall t \geq 0$ . The initial state  $x(0)$  is unknown  $x(0)=?$

The output signal  $y(t)$  is measured and is known. Unfortunately dimension  $m < n$  and the matrix  $C$  is rectangular (less equations than unknown variables). Hence based on single measurement of output vector  $y(t_i)$ , the state vector  $x(t_i)$  cannot be calculated.

Standard formula for the output  $y(t)$  of the above LTI system is

$$y(t) = C e^{At} x(0) \tag{2}$$

Multiplying the both sides of (2) by transposition of the suitable matrix one can obtained

$$e^{A^t t} C^t y(t) = e^{A^t t} C^t C e^{At} x(0) \tag{3}$$

Obtained matrix  $e^{A^t t} C^t C e^{At}$  is square however, still singular for any  $t$ .

Integration of (3) in interval  $[0, T]$  enables calculation of  $x(0)$  if and only if the square Gram matrix  $M_0$  is non-singular and the history of the output signal  $y(t)$  on this interval is known.

$$x(0) = M_0^{-1} \int_0^T e^{A^t t} C^t y(t) dt \tag{4}$$

$$\text{where } M_0 = \int_0^T e^{A^t t} C^t C e^{At} dt \tag{5}$$

Nonsingularity of  $M_0$  is the well-known necessary and sufficient condition for the system, to be state observable. The equivalent algebraic formula for the state observability has the form

$$\text{rank } Q_o = \text{rank} \begin{bmatrix} C \\ C A \\ \vdots \\ C A^{n-m} \end{bmatrix} = n, \text{ for } 1 \leq m \leq n \tag{6}$$

From this equivalence, it is easy to see that for continuous systems the state observability does not depend on the time observation  $T$ .

### The standard Kalman Filter formula used in control tasks

A stationary continuous system in the presence of input noise (or condition) and output noise of the signal is given. It is assumed that an expected value (average) of the initial conditions are known or are zeros.

$$\begin{aligned} \dot{x}(t) &= A x(t) + B u(t) + w(t) \\ y(t) &= C x(t) + v(t) \end{aligned}, \quad x(0) = x_0 \tag{7}$$

The signals  $w(t)$ ,  $v(t)$  are white Gaussian noise with an average value zero:  $E\{w(t)\} = 0$ ,  $E\{v(t)\} = 0$  and the covariance matrices  $Q \geq 0$ ,  $S \geq 0$ ,  $R > 0$  ( $R > 0$  to avoid the singularity of the task):

$$E \left\{ \begin{bmatrix} w(t_1) \\ v(t_1) \end{bmatrix} \begin{bmatrix} w^T(t_2) & v^T(t_2) \end{bmatrix} \right\} = \begin{bmatrix} Q(t_1) & S(t_1) \\ S(t_1) & R(t_1) \end{bmatrix} \delta$$

The task is to find the optimal filter, which for  $\forall t$  will give the state estimate and will guarantee the best filtering of the noise of this state estimate  $\bar{x}(t)$ . The structure of the filter has to be a differential equation similar to the system model and additionally an output measurement for the correction of filtering quality is used.

$$\dot{\bar{x}}(t) = A \bar{x}(t) + B u(t) + G(t) [y(t) - C \bar{x}(t)], \quad \bar{x}(0) = \bar{x}_0 \tag{8}$$

Then the estimation error fulfills the differential equation

$$\dot{\varepsilon}(t) - \hat{\varepsilon}(t) = A [\varepsilon(t) - \bar{\varepsilon}(t)] - G(t) C [\varepsilon(t) - \bar{\varepsilon}(t)] + w(t) - G(t) v(t), \quad x(0) - \bar{x}(0) = \varepsilon(0)$$

$$\dot{\varepsilon}(t) = [A - G(t)C] \varepsilon(t) + [I, -G(t)] \begin{bmatrix} w(t) \\ v(t) \end{bmatrix}, \quad x(0) - \bar{x}(0) = \varepsilon(0) \tag{9}$$

If one assumes that matrix  $G(t) = G$  is constant for

$t \rightarrow \infty$ , and denoting  $F = A - GC$  and also assuming that  $F$  will have eigenvalues with negative real parts, then the solution of the homogeneous equation error (9) (without the noise) will converge to zero.

$$\varepsilon(t) = e^{Ft} \varepsilon(0), \quad \lim_{t \rightarrow \infty} \varepsilon(t) \rightarrow 0$$

The final form of the Kalman filter is:

$$\dot{\bar{x}}(t) = F \bar{x}(t) + B u(t) + G(t) y(t) \tag{10}$$

The main question, however, is what is the form of  $G(t)$ , which for each  $t < \infty$  would additionally guarantee the smallest mean square error of the estimated state.

One can assume that initial conditions  $x(0)$  are a random variable independent from  $w(t)$  and  $v(t)$  with mean value  $E\{x(0)\} = \tilde{x}(0)$  and the variance  $E\{[x(0) - \tilde{x}(0)][x(0) - \tilde{x}(0)]^T\} = Q_0$ .

Current estimation error is given by the equation  $\varepsilon(t) = x(t) - \bar{x}(t)$ .

Average error value and error variance matrix:

$$E\{\varepsilon(t)\} = \tilde{\varepsilon}(t), \quad E\{[\varepsilon(t_1) - \tilde{\varepsilon}(t_1)][\varepsilon(t_1) - \tilde{\varepsilon}(t_1)]^T\} = P(t_1)$$

The matrix of the second order moments

$$E\{\varepsilon(t) \varepsilon^T(t)\} = \tilde{\varepsilon}(t) \tilde{\varepsilon}^T(t) + P(t).$$

The problem is to find a formula for the filter amplification matrix  $G(t)$ , which will guarantee the smallest mean square error, i.e. the minimum of quality index for the given model

$$\min_{\substack{J \\ G}} J = \min_{\substack{G \\ G}} E\{\varepsilon^T(t) \cdot \varepsilon(t)\}, \quad E\{\varepsilon^T(t) \cdot \varepsilon(t)\} = \tilde{\varepsilon}^T(t) \tilde{\varepsilon}(t) + \text{trace}[P(t)]$$

The first part of the last equation reaches the minimum for  $\tilde{\varepsilon}(t) = 0$ . This can be achieved by assuming  $\tilde{\varepsilon}(0) = 0$  because the actual initial condition is known  $\bar{x}(0) = \tilde{x}(0)$ .

Then the error variance matrix equals:

$$P(t) = E\{\varepsilon(t) \cdot \varepsilon^T(t)\}$$

Formula for  $J$  has the form:

$$J = E\{\varepsilon^T(t) \cdot \varepsilon(t)\} = \text{trace}[P(t)]$$

The  $J$  performance index reaches the minimum for the optimal matrix of amplification coefficients, which is non-stationary  $G(t)$  and is expressed by the formula, [5]

$$G(t) = [P(t) \cdot C^T + S] \cdot R^{-1}$$

The square matrix  $P(t)$  satisfies Riccati nonlinear differential equation (11)

$$\dot{P}(t) = AP(t) + P(t)A^T - G(t)RG^T(t) + Q \tag{11}$$

For the case of uncorrelated noise  $S = 0$ , the gain matrix  $G(t)$  and Riccati equation are:

$$G(t) = P(t) \cdot C^T \cdot R^{-1} \tag{12}$$

$$\dot{P}(t) = AP(t) + P(t)A^T - P(t)C^T R^{-1} C P^T(t) + Q \tag{13}$$

with the initial condition  $P(0) = P_0$ .

The solution of the Riccati differential equation is the matrix  $P(t)$ , giving a time variable gain matrix of the filter  $G(t) [n \times m]$ . The optimal Kalman filter is therefore a time-variant (non-stationary) filter. Assuming a positive definite (semi-definite) of  $P(0)$  and the observability of the matrix pair  $(A, C)$ , the solution  $P(t)$  asymptotically tends to the only positively defined (semi-defined) constant matrix  $P(t) [n \times m]$ , for time  $t \rightarrow \infty$ . The method of solving Riccati equation one can find in [6, 7].

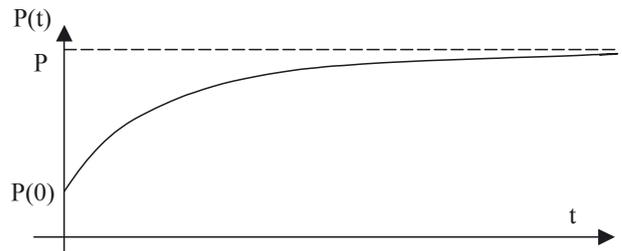


Figure 4. The Riccati equation solution for matrix  $P(t)$

The constant matrix  $P$  is also a solution of the Algebraic Riccati Equation (ARE), (14).

$$0 = AP + PA^T - PC^T R^{-1} C P^T + Q \tag{14}$$

Using the constant matrix  $P$  one obtains a constant gain filter matrix  $G$  (time-invariant filter), which is for  $t < \infty$  suboptimal, only. The Riccati equations one can solve off-line and use the

$G(t)$  or  $G$  in the on-line real-time estimation problems.

**Example 1.**

Given LTI system of the second order (double integrator) with uncorrelated noises

$$\begin{aligned} \dot{x}(t) &= \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t) + \begin{bmatrix} 0 \\ \gamma \end{bmatrix} w(t) \\ y(t) &= \begin{bmatrix} 2 & 0 \end{bmatrix} x(t) + v(t) \end{aligned} \tag{15}$$

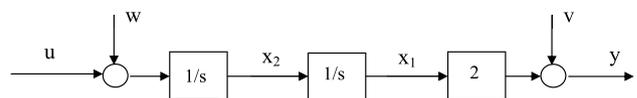


Figure 5. Observed system

The variance of the noise  $w(t)$  is denoted by  $Q$ , the variance of the noise  $v(t)$  by  $R = r$  and the symmetric matrix  $P$  represents the variance of the error:

$$Q = \begin{bmatrix} 0 \\ \gamma \end{bmatrix} [0 \ \gamma] w_1 = \begin{bmatrix} 0 & 0 \\ 0 & \gamma^2 w_1 \end{bmatrix}, \quad P = \begin{bmatrix} p_1 & p_2 \\ p_2 & p_4 \end{bmatrix} = P^T$$

The ARE equation has a form:

$$0 = AP + PA^T + Q - PC^T R^{-1} CP$$

$$0 = \begin{bmatrix} p_2 & p_4 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} p_2 & 0 \\ p_4 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & \gamma^2 w_1 \end{bmatrix} - \frac{4}{r} \begin{bmatrix} p_1^2 & p_1 p_2 \\ p_1 p_2 & p_2^2 \end{bmatrix}$$

$$\left. \begin{aligned} 2p_2 - \frac{4}{r} p_1^2 = 0 \\ p_4 - \frac{4}{r} p_1 p_2 = 0 \\ \gamma^2 w_1 - \frac{4}{r} p_2^2 = 0 \end{aligned} \right\} \text{the solution has a form } \begin{aligned} p_2 &= \frac{\beta}{2} r \\ p_1 &= \frac{1}{2} \sqrt{\beta} r \quad \text{where } \beta = \gamma \sqrt{\frac{w_1}{r}} \\ p_4 &= \beta \sqrt{\beta} r \end{aligned}$$

$$P = \begin{bmatrix} \sqrt{\beta}/2 & \beta/2 \\ \beta/2 & \beta\sqrt{\beta} \end{bmatrix} r, \quad G = PC^T R^{-1} = \begin{bmatrix} \sqrt{\beta} \\ \beta \end{bmatrix}$$

The state observer matrix  $F$  has the form:

$$F = A - GC = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} 2\sqrt{\beta} & 0 \\ 2\beta & 0 \end{bmatrix} = \begin{bmatrix} -2\sqrt{\beta} & 1 \\ -2\beta & 0 \end{bmatrix} \quad (16)$$

Characteristic polynomial and elements of matrix  $F$  depend on the noise stochastic properties

$$s^2 + 2\sqrt{\beta} s + 2\beta = 0$$

$$s_{1,2} = -\sqrt{\beta} \pm j\sqrt{\beta}$$

$$\dot{\bar{x}}(t) = (A - GC) \bar{x}(t) + Bu(t) + Gy(t), \quad (17)$$

$$\dot{\bar{x}}(t) = \begin{bmatrix} -2\sqrt{\beta} & 1 \\ -2\beta & 0 \end{bmatrix} \bar{x}(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t) + \begin{bmatrix} \sqrt{\beta} \\ \beta \end{bmatrix} y(t)$$

For  $\beta = 1$  the eigenvalues and the optimal form of the estimator are:  $s_{12} = -1 \pm j$ .

$$\dot{\bar{x}}(t) = \begin{bmatrix} -2 & 1 \\ -2 & 0 \end{bmatrix} \bar{x}(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t) + \begin{bmatrix} 1 \\ 1 \end{bmatrix} y(t) \quad (18)$$

In the Figure 7 one can see the quality of the state estimation for system (15) by the filter (18) under noisy measurement. In the Figure 8, one can see the quality of the state stabilization for (15) by the use of linear state controller LQR, which generates the control  $u(t) = -K x(t)$  (the gain  $K$  is optimal from the point of view of quadratic quality index) in the case of direct measurement of the system vector state  $x(t)$ . The comparison of the quality of the state stabilization for (15) by the use of linear state controller LQR and the Kalman Filter (18) in the case that the state is not available for measurement, under noisy measurement, is visible on the Figure 9.

### Luenberger deterministic state observer

Given controlled undisturbed system with unknown initial condition

$$\dot{x}(t) = Ax(t) + Bu(t), \quad x(0) = x_0$$

$$y(t) = Cx(t)$$

$$x(t) \in R^n, \quad u(t) \in R^r, \quad y(t) \in R^m, \quad m < n$$

The asymptotic deterministic and stationary (matrix  $G$  constant) state observer has the form of a differential equation like in KF (10).

$$\dot{\bar{x}}(t) = F \bar{x}(t) + Bu(t) + G y(t), \quad \bar{x}(0) = \bar{x}_0$$

Its solution  $\bar{x}(t)$ , starting from the arbitrarily chosen initial state estimate  $\bar{x}(0)$ , follows with the real state  $x(t)$  and converges to it asymptotically if the matrix  $F$  is asymptotically stable and equal to  $F = A - GC$ . The estimation error decreases to zero, but its value is not known.

$$\lim_{t \rightarrow \infty} \varepsilon(t) \rightarrow 0$$

The observer  $G$  gain coefficients are selected by the  $F$  matrix eigenvalues location method. If the system is observable, then for chosen eigenvalues appropriate coefficients of matrix  $G$  can always be found. General analytical formula for the MISO system (scalar  $y \in R$ ) and the column matrix of gain coefficients  $G [n \times 1]$ , gives the Ackermann formula [8].

Denoting desired form of the characteristic polynomial of the observer matrix  $F$  with the given eigenvalues  $s_i$ , the given coefficients  $\alpha_i$  of this polynomial can also be obtained.

$$\alpha(s) = |sI - A + GC| = (s - s_1)(s - s_2) \cdots (s - s_n) = s^n + \alpha_{n-1}s^{n-1} + \dots + \alpha_1s + \alpha_0$$

The corresponding form of the matrix polynomial (matrix  $A$  instead of  $s$ ) has the form:

$$\alpha(A) = A^n + \alpha_{n-1}A^{n-1} + \dots + \alpha_1A + \alpha_0I$$

This form is used in the Ackermann formula for matrix coefficients  $G(t) [n \times 1]$  of a full rank observer, with the use of the observability matrix  $Q_o$ , (6):

$$G = \alpha(A) \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}, \quad (19)$$

### Example 2.

For the deterministic second order LTI system, similar to (15) one should find full order Luenberger observers.

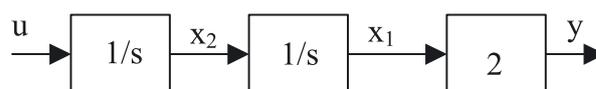


Figure 6. Observable 2nd rank system

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}; \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}; \quad C = [2, \quad 0],$$

The gain matrix  $G$  of the second order observer and the state matrix  $F$  have the form:

$$G = \begin{bmatrix} g_1 \\ g_2 \end{bmatrix}; \quad F = A - GC = \begin{bmatrix} -2g_1 & 1 \\ -2g_2 & 0 \end{bmatrix}; \quad \det(sI - F) = s^2 + 2g_1s + 2g_2 = 0$$

$$s_{1,2} = -g_1 \pm \sqrt{g_1^2 - 2g_2}$$

For assumed  $s_1 = -2$  i  $s_2 = -4$  from the above formulas,  $g_1 = 3$  i  $g_2 = 4$  are obtained. The same values one can find from the Ackerman formula. The observer has the form:

$$\dot{\bar{x}}(t) = \begin{bmatrix} -6 & 1 \\ -8 & 0 \end{bmatrix} \bar{x}(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t) + \begin{bmatrix} 3 \\ 4 \end{bmatrix} y(t) \quad (20)$$

Equivalent Luenberger observer, with reduced first order can have an example form

$$\dot{z}(t) = -2z(t) + u(t) - 2y(t)$$

$$\bar{x}(t) = \begin{bmatrix} 0 \\ 1 \end{bmatrix} z(t) + \begin{bmatrix} 0.5 \\ 1 \end{bmatrix} y(t)$$

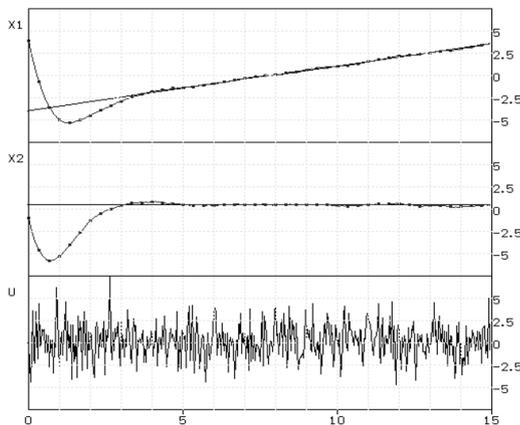


Figure 7. Estimation of the state of (15)

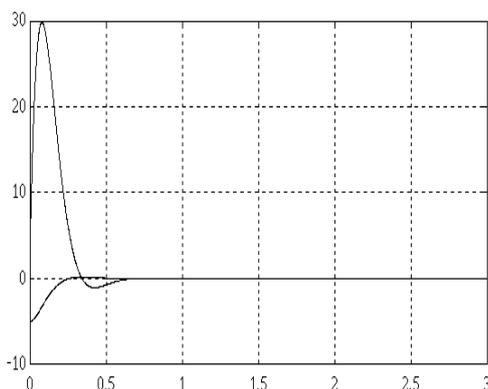


Figure 8. Stabilization of system with LQR with the noise by Kalman Filter. and direct measurement of the state.  $x_1(t)$  and  $x_2(t)$  (without the observer)

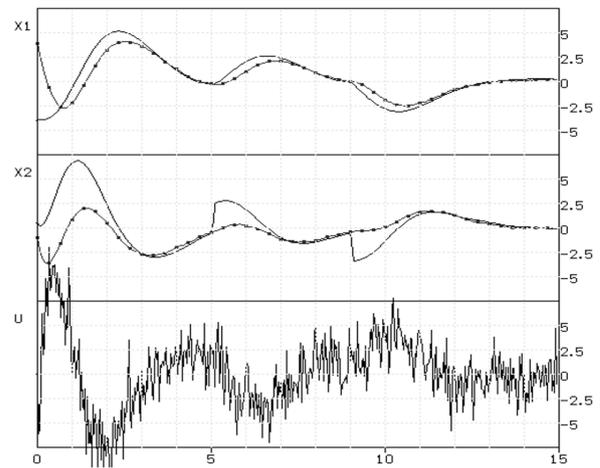


Figure 9. Stabilization with LQR and KF, with measurement noise

### Discrete versions of the Kalman Filter

In many applications of computer control systems very often the plant is modeled by the discrete difference state equations. Below two versions of discrete Kalman Filter will be quoted for more general case of time-depended system matrices (21) and non-unite gain matrix  $W_i$  of the control noise, however, this time the noises  $w$  and  $v$  will be uncorrelated (the covariance matrix  $S = 0$ ). The non-stationary version of the state equation is:

$$\begin{aligned} x_{i+1} &= A_{Di}x_i + B_{Di}u_i + W_iw_i \\ y_i &= C_{Di}x_i + v_i \end{aligned} \quad (21)$$

In the sequel, for increasing the readability of the patterns, the lower index  $D$  will be omitted  $A_{Di} = A_i$ .

The version of Kalman one-step predictor working on the assumption that the predicted state estimate  $\bar{x}_{i+1} = \bar{x}(i+1 | i)$  depends on the current measurement of output  $y_i$  and will not be corrected by the next measurement  $y_{i+1}$ .

Gain matrix:

$$G_i = A_i P_i C_i^T [C_i P_i C_i^T + R_i]^{-1} \quad ; \quad P(0) = P_0$$

State prediction:

$$\bar{x}_{i+1} = [A_i - G_i C_i] \bar{x}_i + B_i u_i + G_i y_i \quad ; \quad \bar{x}(0) = \bar{x}_0 \quad (22)$$

The variance matrix:

$$P_{i+1} = A_i P_i A_i^T - G_i C_i P_i A_i^T + W_i Q_i W_i^T$$

This above basic version of KF is called the prediction filter, and is used in all control problems with the state observation and feedback applications.

The version of the two-step Kalman filter (predictor + corrector) working on the assumption that the predicted state estimate  $\hat{x}_{i+1} = \hat{x}(i+1 | i+1)$  depends on the current measurement of output  $y_i$  and will be additionally corrected with a future measurement of  $y_{i+1}$ . The predicted value of the state, based only on the current output measurement (without correction), as in ver-

sion 1, will be denoted by  $\bar{x}_{i+1} = \bar{x}(i+1|i)$ .

We will denote:

the state prediction:  $\bar{x}_{i+1} = \bar{x}(i+1|i)$ ,

the corrected state prediction:  $\hat{x}_{i+1} = \hat{x}(i+1|i+1)$ .

The steps in KF algorithm:

State prediction:

$$\bar{x}_{i+1} = A_i \hat{x}_i + B_i u_i \quad \hat{x}(0) = x_0$$

Prediction of variance:

$$\bar{P}_{i+1} = A_i \hat{P}_i A_i^T + W_i Q_i W_i^T$$

Innovation (residuum):

$$e_{i+1} = y_{i+1} - \bar{y}_{i+1} = y_{i+1} - C_{i+1} \bar{x}_{i+1}$$

The variance of innovation:

$$L_{i+1} = C_{i+1} \bar{P}_{i+1} C_{i+1}^T + R_{i+1}$$

Filter gain:

$$G_{i+1} = \bar{P}_{i+1} C_{i+1}^T L_{i+1}^{-1} = \bar{P}_{i+1} C_{i+1}^T [C_{i+1} \bar{P}_{i+1} C_{i+1}^T + R_{i+1}]^{-1}$$

The variance correction:

$$\hat{P}_{i+1} = [I - G_{i+1} C_{i+1}] \bar{P}_{i+1}$$

Corrected state prediction, that is, the filter equation:

$$\begin{aligned} \hat{x}_{i+1} &= \bar{x}_{i+1} + G_{i+1} e_{i+1} = [I - G_{i+1} C_{i+1}] \bar{x}_{i+1} + G_{i+1} y_{i+1} = \\ &= A_i \hat{x}_i + B_i u_i + G_{i+1} e_{i+1}. \end{aligned} \quad (23)$$

The above version is usually called the Kalman Filter.

It is worth noting that in on-line control applications in which the regulator generates the control signal in the current step, for the next sampling period based on the current output measurement and one-step prediction of the state based on the equation  $u_{i+1} = -K \bar{x}_{i+1}$ , then only the Kalman predictor, can be used formally, because only the predicted state  $\bar{x}_{i+1}$  and the control  $u_{i+1}$  can generate a future real state  $x_{i+1}$  and a real output  $y_{i+1} = f(u_{i+1})$ , that can no longer be used for correction of the used state  $\bar{x}_{i+1}$ .

The use in the Kalman filter, the predicted state  $\hat{x}_{i+1}$  corrected by  $y_{i+1}$ , for control generation in the same sample, is not realizable in the feedback loop. Therefore, the discrete Kalman filter (in the predictor-corrector version) is most often used for the best continuous filtration of measurement signals (e.g. video or audio) without the possibility of using this signal in the feedback loop.

For decreasing sampling of times, both of these versions discussed above converge to one common version, which is equivalent to a continuous version of KF. For nonlinear systems, the so-called Extended Kalman Filter was invented, which uses data from a non-linear model in every step, and in the algorithm, it uses the matrix of the linearized model [9].

## Conclusions

In this paper, the theory of the optimal asymptotic state filter and asymptotic state observer was recalled and presented. Three versions of Kalman Filter were compared: the continuous ver-

sion of KF, one-step predictor KF and two-step Kalman filter (predictor + corrector). For control processes very often the simplified deterministic version of Kalman Filter is used, so-called Luenberger state observer. Numerical examples show that the state estimation and the state stabilization by the use of Kalman Filter are characterized by the asymptotic convergence to the real state functions. Changing the covariance matrices  $Q$  and  $R$  the filter gain  $G$  is changing. The dynamics and quality of the Luenberger observer depend on the observer gain, which is calculated based on pole placement technique. The asymptotic behavior of the convergence of the state estimate given by KF or Luenberger observer is due to their structure based on linear differential/difference equation.

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