On the homogeneous extremal function for the standard simplex

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Abstract
In this paper an explicit formula for the homogeneous Siciak's extremal function is computed in the case of standard simplex in R^N. There are discussed some problems related to this result. In particular, there is proved a version of Klimek's type theorem for the homogeneous extremal function.

Key words: Homogeneous extremal function, convex stets

1. Introduction

Let $\mathcal{P}(\mathbb{C}^N)$ and $\mathcal{H}(\mathbb{C}^N)$ denotes the space of polynomials or the space of homogeneous polynomials of N complex variables, respectively. $PSH(\Omega)$ is the cone of plurisubharmonic functions on Ω. Let us recall definitions of two important extremal functions associated to a compact E in $\mathbb{C}^N$ introduced by Józef Siciak in the sixties of the twentieth century.

$$\Phi(E,z) = \sup\{|P(z)|^{1/\deg P}:\ P \in \mathcal{P}(\mathbb{C}^N),\ \deg P \geq 1, ||P||_E \leq 1\},\ z \in \mathbb{C}^N,$$

$$\Psi(E,z) = \sup\{|P(z)|^{1/\deg P}:\ P \in \mathcal{H}(\mathbb{C}^N),\ \deg P \geq 1, ||P||_E \leq 1\},\ z \in \mathbb{C}^N.$$

The basic property of those functions is related to an important fact that they can be obtained by large families of plurisubharmonic functions, see [7],[8]. To descript this, let us recall that the Lelong class $L(\mathbb{C}^N)$ is equal to the family $u \in PSH(\mathbb{C}^N)$ such that $\sup\{u(z) - \log(1 + (|z_1|^2 + \cdots + |z_N|^2)^{1/2})\} < \infty$. Another important family is $H(\mathbb{C}^N) = \{f \in PSH(\mathbb{C}^N) : f \not\equiv 0,\ f(\lambda z) = |\lambda| f(z),\ \lambda \in \mathbb{C}, z \in \mathbb{C}^N\}$.

Theorem 1.1. *(a) (Zakharyuta-Siciak theorem)*

$$\Phi(E,z) = \exp V(E,z) = \exp\sup\{u(z):\ u \in L(\mathbb{C}^N),\ u|_E \leq 0\},\ z \in \mathbb{C}^N.$$

b) (Siciak's theorem)

$$\Psi(E,z) = \sup\{f(z):\ f \in H(\mathbb{C}^N),\ f|_E \leq 1\}.$$

Zakharyuta-Siciak theorem is a basic link between approximation theory and pluripotential methods, we refer to [5] for details.

Remark 1.2. Let us recall a few important properties of the above extremal functions.

a) $\Phi(E,z) \leq \Phi(F,z)$ and $\Psi(E,z) \leq \Psi(F,z)$ in the case $F \subset E$.

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b) If $E_n \searrow E$ then $\Phi(E_n, z) \nearrow \Phi(E, z)$ and $\Psi(E_n, z) \nearrow \Psi(E, z)$.
c) If E is a circular set then $\Phi(E, z) = \max(1, \Psi(E, z)),\ z \in \mathbb{C}^N$.
d) Let $L(z) = \widehat{L}(z) + b$, where $b \in \mathbb{C}^N$ and $\widehat{L}$ is a linear isomorphism of $\mathbb{C}^N$. Then for all $z \in \mathbb{C}^N$

$$\Phi(L^{-1}(E), z) = \Phi(E, L(z)), \tag{1.1}$$

$$\Psi(\widehat{L}^{-1}(E), z) = \Phi(E, \widehat{L}(z)). \tag{1.2}$$

We shall see in the last section that in (1.2) we can not replace $\widehat{L}(z)$ by $L(z)$. On the other hand (1.1) can be generalized to the more general case of polynomial mappings by well known Klimek's theorem. Also in (1.2) linear isomorphisms $\widehat{L}(z)$ can be replaced by a more general class of homogeneous polynomial mappings.

Example 1.3. If $E = \mathbb{B}_N = \{z \in \mathbb{C}^N :\ z_j \in \mathbb{R},\ z_1^2 + \cdots + z_N^2 \leq 1\}$ then by the Lundin formula (cf. [5] where the proof is presented) we can write

$$\Phi(E, z) = h(|z_1|^2 + \cdots + |z_N|^2 + |z_1^2 + \cdots + z_N^2 - 1|)^{1/2},\ z \in \mathbb{C}^N,$$

where $h(t) = t + \sqrt{t^2 - 1}$ for $t \geq 1$. Applying methods from [1],[2] one can obtain (cf. [4]) the following interesting formula

$$\Phi(S^{N-1}, z) = \Phi(\mathbb{B}_N, z) = h(|z_1|^2 + \cdots + |z_N|^2)^{1/2},\ \text{if } z_1^2 + \cdots + z_N^2 = 1.$$

Example 1.4.

(1) If E is the closed unit ball in $\mathbb{C}^N$ with respect to a norm $||z||$ then there is well known that $\Psi(E, z) = ||z||$ (while $\Phi(E, z) = \max(1, ||z||)$ and thus $\Phi(E, z) = \Psi(z, E),\ ||z|| \geq 1$).

(2) A situation is much more complicated if E is a convex symmetric body in $\mathbb{R}^N$. There was known in the case $E = \mathbb{B}_N$ that $\Psi(E, z) = \Psi(S^{N-1}, z) = L_N(z)$ is the Lie norm in $\mathbb{C}^N$ and $\Phi(S^{N-1}, z) = \Psi(S^{N-1}, z)$ for $z \in \mathbb{S}_{N-1}$, where

$$S^{N-1} = \{x \in \mathbb{R}^N :\ x_1^2 + \cdots + x_N^2 = 1\} \subset \mathbb{S}^{N-1} = \{z \in \mathbb{C}^N :\ z_1^2 + \cdots + z_N^2 = 1\}.$$

If $N > 2$ a situation is quite unclear. But in the case $N = 2$ there is known the following result (cf. [3]):

Let S be the unit ball with respect to a norm N in $\mathbb{R}^2$. If $u(t) = \log N(1, t)$ then

$$\Psi(S, (z_1, z_2)) = |z_1| \exp \mathcal{P}u(z_2/z_1),$$

with

$$\mathcal{P}u(\zeta) = (\Im\zeta)\frac{1}{\pi}\int_{-\infty}^{\infty} |\zeta - t|^{-2} u(t)dt = \frac{1}{\pi}\int_{-\infty}^{\infty} u(ty + x)\frac{dt}{1 + t^2},$$

where $\zeta = x + iy,\ y \geq 0$. In particular, if

$$N_m(x) = (|x_1|^m + |x_2|^m)^{1/m} \quad \text{and } \mathcal{S}_m = \{x \in \mathbb{R}^2 : \ N_m(x) = 1\},$$

then for all $z \in \mathbb{C}^2$,

$$\Psi(\mathcal{S}_{2m}, z) = \left[\prod_{j=1}^{m} \left(|z_1|^2 - 2\alpha_j \Re(z_1\overline{z_2}) + |z_2|^2 + 2|\beta_j|\Im(z_1\overline{z_2})|\right)^{1/2}\right]^{1/m},$$

where $\zeta_j = \alpha_j + i\beta_j \in \sqrt[2m]{-1},\ j = 1, \ldots, m$, with $\zeta_j \neq \overline{\zeta_k}$ for $j \neq k$.

If $N_\infty(x) = \max(|x_1|, |x_2|)$ and $\mathcal{S}_\infty = \{x \in \mathbb{R}^2 : \ N_\infty(x) = 1\}$, then for all $z \in \mathbb{C}^2$,

$$\log \Psi(\mathcal{S}_\infty, z) = \int_0^{2\pi} \log \left(|z_1|^2 - 2\cos\theta \Re(z_1\overline{z_2}) + |z_2|^2 + 2|\sin\theta \Im(z_1\overline{z_2})|\right)^{1/2} \frac{d\theta}{2\pi}.$$

Since $\mathcal{S}_1 = \{x \in \mathbb{R}^2 : \ |x_1| + |x_2| = 1\} = L^{-1}(S_\infty)$, where $L(z_1, z_2) = (z_1 - z_2, z_1 + z_2)$, we get

$$\log \Psi(\mathcal{S}_1, z) = \log \Psi(\mathcal{S}_\infty, L(z))$$

$$= \int_0^{2\pi} \log \left(2|z_1|^2 + 2|z_2|^2 - 2\cos\theta(|z_1|^2 - |z_2|^2) + 4|\sin\theta \Im(z_1\overline{z_2})|\right) \frac{d\theta}{4\pi}.$$

Let us note, that except $m = 2$ the homogeneous extremal function $\Psi(E, (z_1, z_2))$ is not a norm in $\mathbb{C}^2$.

Let us go to the mentioned case of the unit Euclidean ball in $\mathbb{R}^N$.

Proposition 1.5. *(Siciak's formula)*

$$\Psi(\mathbb{B}_N, z) = L_N(z) = \left(\frac{||z||_2^2 - |z^2|}{2}\right)^{1/2} + \left(\frac{||z||_2^2 + |z^2|}{2}\right)^{1/2}$$

$$= \left(|z_1|^2 + \cdots + |z_N|^2 + \left((|z_1|^2 + \cdots + |z_N|^2)^2 - |z_1^2 + \cdots + z_N^2|^2\right)^{1/2}\right)^{1/2}.$$

Here $||z||_2^2 = |z_1|^2 + \cdots + |z_N|^2$, $z^2 = z_1^2 + \cdots + z_N^2$. *In the special case* $N = 2$ *we have*

$$\Psi(\mathbb{B}_2, z) = \max(|z_1 - iz_2|, |z_1 + iz_2|).$$

Since $\log L_N \in PSH(\mathbb{C}^N)$, we get $L_N^2 \in PSH(\mathbb{C}^N)$. Define

$$u_N(z) = \max\{L_N^2(w_1, \ldots, w_N) : \ w_1^2 = z_1, \ldots, z_N^2 = z_N\}$$

$$= |z_1| + \cdots + |z_N| + \left((|z_1| + \cdots + |z_N|)^2 - |z_1 + \cdots + z_N|^2\right)^{1/2}.$$

By Proposition 2.9.26 in [5] we deduce that $u_N \in PSH(\mathbb{C}^N)$.

In the special case $N = 2$ we can apply another argument (which fails if $N \geq 3$). Namely, one can easily calculate that $\left[\frac{\partial^2 v_2}{\partial z_j \partial \overline{z_k}}\right] = 0$ on $\mathbb{C}^N \setminus F$, where $v_2(z_1, z_2) = (2|z_1\overline{z_2}| - \Re(z_1\overline{z_2}))^{1/2}$ and $F = \{z \in \mathbb{C}^2 : \ z_1\overline{z_2} \geq 0\}$ is a subset of Lebesgue measure equals 0. Since v_2 is continuous we obtain $v_2 \in PSH(\mathbb{C}^2)$ and therefore $u_2(z) = |z_1| + |z_2| + v_2(z) \in PSH(\mathbb{C}^2)$.

We also have $\left[\frac{\partial^2 v_2}{\partial z_j \partial \overline{z_k}}\right] = 0$ on $\mathbb{C}^2$ in the distributional sense,$(dd^c)^2 v_2 = 0$ and $(dd^c)^2 u_2 = (dd^c)^2(|z_1| + |z_2|)$.

2. Homogeneous extremal function for the simplex

Actually there is well known an explicit formula ([1],[2], see also [5]) for the Siciak's extremal function in the case of standard simplex

$\widetilde{S_N} = \{x \in \mathbb{R}^N : \ x_1 \geq 0, \dots, x_N \geq 0, x_1 + \cdots + x_N \leq 1\} = \mathrm{conv}(0 \cup S_{N-1})$,

where

$$S_{N-1} = \{x \in \mathbb{R}^N : \ x_1 \geq 0, \dots, x_N \geq 0, x_1 + \cdots + x_N = 1\} \subset \mathbb{S}_{N-1}$$
$$= \{z \in \mathbb{C}^N : \ z_1 + \cdots + z_N = 1\}.$$
$$\Phi(\widetilde{S_N}, z) = h(|z_1| + \cdots + |z_N| + |z_1 + \cdots + z_N - 1|).$$

Hence for an arbitrary $\zeta \in \mathbb{C} \setminus \{0\}$

$$\Psi(S_{N-1}, z) = \Psi(\widetilde{S_N}, z) \leq |\zeta| \Phi(\widetilde{S_N}, \zeta^{-1} z)$$
$$= |\zeta| h(|z_1/\zeta| + \cdots + |z_N/\zeta| + |\zeta^{-1}(z_1 + \cdots + z_N) - 1|).$$

In particular, taking $\zeta = z_1 + \cdots + z_N \neq 0$ we get

$$\Psi(S_{N-1}, z) \leq |z_1 + \cdots + z_N| h(|z_1/\zeta| + \cdots + |z_N/\zeta|)$$
$$= |z_1| + \cdots + |z_N| + \left((|z_1| + \cdots + |z_N|)^2 - |z_1 + \cdots + z_N|^2\right)^{1/2}.$$

Theorem 2.1. *For an arbitrary $N \geq 1$ and for all $z \in \mathbb{C}^N$ we have equality*

$$\Psi(S_{N-1}, z) = |z_1| + \cdots + |z_N| + \left((|z_1| + \cdots + |z_N|)^2 - |z_1 + \cdots + z_N|^2\right)^{1/2}$$
$$= |z_1| + \cdots + |z_N| + 2\left(\sum_{1 \leq i < j \leq N} (\Im \sqrt{z_i \overline{z_j}})^2\right)^{1/2}$$

Proof. We have inequalities

$$\Psi(S_{N-1}, z) = \Psi(\widetilde{S_N}, z)$$
$$\leq |z_1| + \cdots + |z_N| + \left((|z_1| + \cdots + |z_N|)^2 - |z_1 + \cdots + z_N|^2\right)^{1/2} = u_N(z).$$

Since $u_N(\lambda z) = |\lambda| u_N(z)$, $u_N \in PSH(\mathbb{C}^N)$ and $u_N | S_{N-1} = 1$ we see that $u_N \in H(\mathbb{C}^N)$. By Siciak's theorem we get inequality $u_N(z) \leq \Psi(S_{N-1}, z)$ which finishes the proof.

□

3. Examples and remarks

We have $\Psi(\mathbb{B}_N, z) = \Phi(\mathbb{B}_N, z) = \sqrt{h(||z||^2)}$ for $z \in \mathbb{S}^{N-1}$. Moreover, as it was proved in [4], we also have

$$\Phi(S^{N-1}, z) = \Phi(\mathbb{B}_N, z) = \Psi(\mathbb{B}_N, z) = \Psi(S^{N-1}, z) = \sqrt{h(||z||^2)},\ z \in z \in \mathbb{S}^{N-1}.$$

Let $F(z) = (z_1^2, \ldots, z_N^2)$. Then $F^{-1}(S_{N-1}) = S^{N-1}$. By Klimek's theorem (cf. [5], Theorem)

$$h(||z||^2) = \Phi(S^{N-1}, z)^2 = \Phi(F^{-1}(S_N), z)^2 = \Phi(S_{N-1}, F(z)),\ z \in \mathbb{S}^{N-1},$$

which implies

$$\Phi(S_{N-1}, z) = h(|z_1| + \cdots + |z_N|) = h(||z||_1),\ z \in \mathbb{S}_{N-1}.$$

Corollary 3.1. *For an arbitrary $N \geq 2$*

$$\Phi(S_{N-1}, z) = \Phi(\widetilde{S_N}, z) = \Psi(\widetilde{S_N}, z) = \Psi(S_{N-1}, z),\ z \in \mathbb{S}_{N-1}.$$

Example 3.2. Let $E = \{x \in \mathbb{R}^2 :\ x_1, x_2 \in [-1/2, 1/2],\ x_1 + x_2 = 0\} = S_1 - (1/2, \ldots, 1/2) \subset \{z :\ z_1 + z_2 = 0\}$. Then

$$\Phi(E, z) = \Phi(S_{N-1}, z + (1/2, 1/2)) = h(|z_1 + 1/2| + |z_2 + 1/2|),\ z_1 + z_2 = 0.$$

$$\Psi(S_1, z + (1/2, 1/2)) =$$

$$|z_1 + 1/2| + |z_2 + 1/2| + \left((|z_1 + 1/2| + |z_2 + 1/2|)^2 - |z_1 + z_2 + 1|^2\right)^{1/2},$$

for all $z \in \mathbb{C}^2$. But

$$\Psi(E, z) = \Psi(S_1 - (1/2, 1, 2), z) = \lim_{n\to\infty} \Psi(E_n, z)$$
$$= \lim_{n\to\infty} \max(|z_1 - z_2 - in(z_1 + z_2)|, |z_1 - z_2 + in(z_1 + z_2)|)$$
$$= \begin{cases} |z_1 - z_2|,\ z_1 + z_2 = 0, \\ +\infty,\ z_1 + z_2 \neq 0, \end{cases}$$

where $E_n = \{x :\ (x_1 - x_2)^2 + n^2(x_1 + x_2)^2 \leq 1\} = \Lambda_n^{-1}(\mathbb{B}_2)$, $\Lambda_n(z) = (z_1 - z_2, n(z_1 + z_2)$. We see that $\Psi(S_1 - (1/2, 1/2), z) \neq \Psi(S_1, z + (1/2, 1/2)$ for all $z \in \mathbb{C}^2$. Moreover, $\Phi(S_1 - (1/2, 1/2), z) = \Psi(S_1 - (1/2, 1/2), z)$ iff $z = (-1/2, 1/2)$ or $z = (1/2, -1, 2)$.

Let us formulate a version of Klimek's theorem for homogeneous polynomial mapping.

Theorem 3.3. *Let $H(z) = (H_1(z), \ldots, H_N(z))$, $H_j \in \mathcal{H}(\mathbb{C}^N)$,* $\deg H_j =$ *$d \geq 1$, $j = 1, \ldots, N$ and $H^{-1}(\{0\}) = \{0\}$. Then for an arbitrary compact $E \subset \mathbb{C}^N$*

$$\Psi(H^{-1}(E), z) = \Psi(E, H(z))^{1/d},\ z \in \mathbb{C}^N.$$

Proof. A proof is a modification of Klimek's proof of his theorem presented in[5].

If $z \in H^{-1}(E)$ then $H(z) \in E$ and therefore

$$||Q \circ H||_{H^{-1}}(E) \le 1 \text{ if } Q \in \mathcal{H}(\mathbb{C}^N),\ ||Q||_E \le 1.$$

Hence we get inequality $\Psi(E, H(z))^{1/d} \le \Psi(H^{-1}(E), z),\ z \in \mathbb{C}^N$. Now consider $Q \in \mathcal{H}(\mathbb{C}^N),\ ||Q||_{H^{-1}}(E) \le 1$ and define

$$f(z) = \sup\{|Q(w)|^{d/\deg P} :\ w \in H^{-1}(z)\},\ z \in \mathbb{C}^N.$$

We now have, again by Proposition 2.9.26 in [5], $f \in PSH(\mathbb{C}^N)$ and

$$f(\lambda z) = \sup\{|Q(w)|^{d/\deg Q} :\ H(w) = \lambda z\}$$

$$= |\lambda| \sup\{|Q(\lambda^{-1/d} w)|^{d/\deg Q} :\ H(\lambda^{-1/d} w) = z\} = |\lambda| f(z)|.$$

Thus $f \in H(\mathbb{C}^N)$ and $f|_E \le 1$ which gives inequality $f(z) \le \Psi(E, z)$, by the Siciak Theorem 1.2 b). Taking the supremum we obtain

$$\sup\{\Psi(H^{-1}(E), w)^d :\ H(w) = z\} \le \Psi(E, z),\ z \in \mathbb{C}^N.$$

In particular $\Psi(H^{-1}(E), w) \le \Psi(E, H(w))^{1/d}$ which finishes the proof. □

Example 3.4. Consider $\widetilde{K_{1/2}} = \{x \in \mathbb{C}^2 :\ x_1, x_2 \ge 0, x_1^{1/2} + x_2^{1/2} \le 1\}$, $K_{1/2} = \{x \in \widetilde{E} :\ x_1^{1/2} + x_2^{1/2} = 1\}$ and $H(z) = \left(\left(\frac{z_1 - z_2}{2}\right)^2, \left(\frac{z_1 + z_2}{2}\right)^2\right)$ with

$$H^{-1}(\widetilde{K_{1/2}}) = \{x :\ |x_1|, |x_2| \le 1\} = \mathrm{conv}(\mathcal{S}_\infty),\ H^{-1}(K_{1/2}) = \mathcal{S}_\infty.$$

Thus applying both versions of Klimek's theorem we obtain

$$\Psi(K_{1/2}, z) = \Psi(\widetilde{K_{1/2}}, z) = \Psi(\mathcal{S}_\infty, (\sqrt{z_1} + \sqrt{z_2}, \sqrt{z_1} - \sqrt{z_2}))$$

$$= \exp \int_0^{2\pi} \log(2|z_1| + 2|z_2| - 2\cos\theta(|z_1| - |z_2|) + 4|\sin\theta \Im(\sqrt{z_1 \overline{z_2}})|) \frac{d\theta}{2\pi},$$

$$\Phi(\widetilde{K_{1/2}}, z) = \max(|h(\sqrt{z_1} + \sqrt{z_2})|, |h(\sqrt{z_1} - \sqrt{z_2})|).$$

Take $\mathbb{K}_{1/2} = \{z \in \mathbb{C}^2 :\ 4z_2 = (1 + z_2 - z_1)^2\} = \{(\zeta^2, (1-\zeta)^2) :\ \zeta \in \mathbb{C}\}$. We have $\Phi(\widetilde{K_{1/2}}, (\zeta^2, (1-\zeta)^2) = |h(2\zeta - 1)|^2 = \Phi([0,1], \zeta)^2$. Moreover, since $\Phi(K_{1/2}, z) \ge \Phi(\widetilde{K_{1/2}}, z)$, we get

$$\Phi(K_{1/2}, (\zeta^2, (1-\zeta)^2)) \ge |h(2\zeta - 1)|^2 = |h(2(2\zeta-1)^2 - 1)| = |h(8\zeta^2 - 8\zeta + 1)|.$$

On the other hand since $u(\zeta) = \log|h(8\zeta^2 - 8\zeta + 1)|$ is a harmonic function on $\mathbb{C} \setminus [0,1]$, equals 0 on $[0,1]$, we deduce that

$$\Phi(K_{1/2}, (\zeta^2, (1-\zeta)^2)) = \Phi(\widetilde{K_{1/2}}, (\zeta^2, (1-\zeta)^2)) = |h(8\zeta^2 - 8\zeta + 1)|,\ \zeta \in \mathbb{C}.$$

In particular, $\Phi(K_{1/2}, z) = \Phi(\widetilde{K_{1/2}}, z)$ for $z \in \mathbb{K}_{1/2}$. Now, if $K_{1/2} \subset E \subset \widetilde{K_{1/2}}$, we also have $\Phi(E, z) = \Phi(K_{1/2}, z)$ for $z \in \mathbb{K}_{1/2}$.

Example 3.5. Let $E_0 = [0,2] \times [0,1] \cap 2\widetilde{S}_2 - (1,0)$. We have

$$\Phi(E_0, z) = \max(\Phi([0,2] \times [0,1], z + (1,0)), \Phi(2\widetilde{S}_2, z + (1,0)))$$
$$= h\left(\max\left(\frac{1}{2}|z_1 + 1| + |\frac{1}{2}z_1 - 1/2|, |z_2| + |z_2 - 1|,\right.\right.$$
$$\left.\left.|\frac{1}{2}(|z_1 + 1| + |z_2|)| + |\frac{1}{2}(z_1 + z_2) - 1/2|\right)\right)$$

and we can easily compute that

$$\Phi(E_0, z) = \Phi(S_1, z) = h(|z_1| + |z_2|),\ z_1 + z_2 = 1.$$

Hence for an arbitrary set $S_1 \subset E \subset E_0$ we also have $\Phi(E, z) = \Phi(S_1, z),\ z_1 + z_2 = 1$.

Remark 3.6. The last two examples are related to the following problem.

Let E_0 be a compact subset of $\mathbb{R}^N$, $E \subset E_0$ and there exist irreducible polynomials $p_1, \ldots, p_s$ such that $E = p_1^{-1}(0) \cap \cdots \cap p_s^{-1}(0) \cap \partial E_0$. Let $\mathbb{E} = \{z \in \mathbb{C}^N :\ p_1(z) = \cdots = p_s(z) = 0\}$. When $\Phi(E, z) = \Phi(E_0, z)$ for $z \in \mathbb{E}$?

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