Superlinear Robin problems with indefinite linear part

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Abstract
We consider a semilinear Robin problem driven by the Laplacian plus an indefinite and unbounded potential and a superlinear reaction term which need not satisfy the Ambrosetti-Rabinowitz condition. Using variational tools we prove two theorems. An existence theorem producing a nontrivial smooth solution and a multiplicity theorem producing a whole unbounded sequence of nontrivial smooth solutions.

Key words: Indefinite and unbounded potential, superlinear reaction, almost critical growth, regularity theory, local linking, infinitely many solutions

1 Introduction

Let $\Omega \subseteq \mathbb{R}^N$ be a bounded domain with a C^2-boundary $\partial\Omega$. In this paper, we study the following semilinear Robin problem:

$$\begin{cases} -\Delta u(z) + \xi(z)u(z) = f(z, u(z)) & \text{in } \Omega, \\ \frac{\partial u}{\partial n} + \beta(z)u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.1}$$

In this problem the potential function $\xi \in L^s(\Omega)$, $s > N$ is in general indefinite (that is, sign changing). The reaction term $f(z, \zeta)$ is a Carathéodory function (that is, for all $\zeta \in \mathbb{R}$, $z \longmapsto f(z, \zeta)$ is measurable and for almost all $z \in \Omega$ $\zeta \longmapsto f(z, \zeta)$ is continuous). We assume that $f(z, \cdot)$ has almost critical growth (so, it does not have in general the usual subcritical growth) and $f(z, \cdot)$ is superlinear but without satisfying the usual in such cases Ambrosetti-Rabinowitz condition. Instead we employ a more general condition which incorporates in our framework superlinear reactions with slower growth near $\pm\infty$ which fail to satisfy the Ambrosetti-Rabinowitz condition. Near zero we assume that $f(z, \cdot)$ is strictly sublinear. In the boundary condition, $\frac{\partial u}{\partial n}$, for $u \in H^1(\Omega)$, stands for the usual normal derivative defined by extension of the continuous linear map

$$C^1(\overline{\Omega}) \ni u \longmapsto \frac{\partial u}{\partial n} = (Du, n)_{\mathbb{R}^N},$$

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with $n(\cdot)$ being the outward unit normal on $\partial\Omega$. The boundary coefficient $\beta \in W^{1,\infty}(\partial\Omega)$ with $\beta(z) \geqslant 0$ for all $z \in \partial\Omega$. When $\beta \equiv 0$, we have the usual Neumann problem.

Recently there have been existence and multiplicity results for semilinear elliptic equations with general potential. We mention the work of Gasiński-Papageorgiou [5], Li-Wang [10], Papageorgiou-Papalini [14], Qin-Tang-Zhang [19] (Dirichlet problems), Papageorgiou-Rădulescu [16], Papageorgiou-Rădulescu [17] (Neumann problem) and Papageorgiou-Smyrlis [18], Shi-Li [21] (Robin problems). Superlinear equations were considered only in the context of Dirichlet problems under more restrictive conditions on the data by Li-Wang [10] and Qin-Tang-Zhang [19]. For other boundary value problems with Robin boundary condition we refer to Bai-Gasiński-Papageorgiou [2], Gasiński-O'Regan-Papageorgiou [3], Gasiński-Papageorgiou [6, 7, 8, 9].

In this paper using variational tools based on the critical point theory, we prove two theorems. The first is an existence theorem producing a nontrivial smooth solution. In the second theorem, under a symmetry condition on $f(z,\cdot)$, we produce an unbounded sequence of nontrivial smooth solutions.

2 Mathematical Background

Let X be a Banach space and let X^* denote its topological dual. By $\langle\cdot,\cdot\rangle$ we denote the duality brackets for the pair (X^*, X). Given $\varphi \in C^1(X;\mathbb{R})$, we say that φ satisfies the $(C)^*$-*condition*, if the following property holds:

> "Every sequence $\{u_n\}_{n\geqslant 1} \subseteq X$ such that $\sup\limits_{n\geqslant 1} \varphi(u_n) < +\infty$ and
>
> $$(1 + \|u_n\|))\varphi'(u_n) \longrightarrow 0 \quad \text{in } X^*,$$
>
> admits a strongly convergent subsequence which converges to a critical point of φ."

Remark 2.1. This is a slightly more general version of the well-known *Cerami condition*, which says that:

> "Every sequence $\{u_n\}_{n\geqslant 1} \subseteq X$ such that $|\varphi(x_n)| \leqslant M$ for some $M > 0$ and all $n \in \mathbb{N}$ and
>
> $$(1 + \|u_n\|)\varphi'(u_n) \longrightarrow 0 \quad \text{in } X^*,$$
>
> admits a strongly convergent subsequence."

This is a compactness type condition on the functional φ more general than the classical Palais-Smale condition. The Cerami condition suffices to have a deformation theorem from which one can derive the minimax theory of the critical values of φ. The Cerami condition and the Palais-Smale condition are equivalent if φ is bounded below (see Motreanu-Motreanu-Papageorgiou [13, p. 104]).

Also, suppose that X admits a direct sum decomposition

$$X = Y \oplus V. \tag{2.1}$$

We say that $\varphi \in C^1(X;\mathbb{R})$ has a local linking at $u = 0$ with respect to (2.1), if there exists $r > 0$ such that

$$\begin{aligned} \varphi(y) &\leqslant 0 \quad \text{for all } y \in Y, \text{ with } \|y\| \leqslant r, \\ \varphi(v) &\geqslant 0 \quad \text{for all } v \in V, \text{ with } \|v\| \leqslant r. \end{aligned}$$

The following existence theorem is due to Luan-Mao [12, Theorem 2.2].

Theorem 2.2. *If $\varphi \in C^1(X;\mathbb{R})$ satisfies the following assumptions:*
(i) φ has a local linking at $u = 0$ with respect to (2.1);
(ii) φ satisfies the $(C)^*$-condition;
(iii) φ maps bounded sets into bounded sets;
(iv) for every finite dimensional subspace Z of V we have

$$\varphi(u) \longrightarrow -\infty \quad \textit{for all } u \in Y \oplus Z, \textit{ with } \|u\| \to +\infty,$$

then φ has at least two critical points.

Remark 2.3. According to the above theorem, φ has at least one nontrivial critical point.

Another result that we will use is the so called "Symmetric Mountain Pass Theorem" of Rabinowitz [20] (see also Gasiński-Papageorgiou [4, p. 688]).

Theorem 2.4. *If X is an infinite dimensional Banach space with a direct sum decomposition*

$$X = Y \oplus E \quad \textit{with } Y \textit{ finite dimensional},$$

$\varphi \in C^1(X;\mathbb{R})$ is even, satisfies the Cerami condition, $\varphi(0) = 0$ and
(i) there exist $\eta, r > 0$ such that

$$\varphi\big|_{E\cap\partial B_r} \geqslant \eta,$$

with $\partial B_r = \{u \in X : \|u\| = r\}$;
(ii) for every finite dimensional subspace $Z \subseteq X$ there exists $\varrho = \varrho(Z) > 0$ such that

$$\varphi\big|_{Z\setminus(Z\cap B_\varrho)} \leqslant 0,$$

with $B_\varrho = \{u \in X : \|u\| < \varrho\}$,
then φ admits an unbounded sequence of critical values.

Next, let us recall some basic facts about the spectrum of $u \longmapsto -\Delta u + \xi(z)u$, $u \in H^1(\Omega)$, with Robin boundary condition. For details see D'Aguì-Marano-Papageorgiou [1].

First we introduce the spaces which we will use in the sequel. These are:

- the Sobolev space $H^1(\Omega)$;
- the Banach sapce $C^1(\overline{\Omega})$;
- the boundary Lebesgue spaces $L^q(\partial\Omega)$ with $1 \leqslant q \leqslant +\infty$.

We know that $H^1(\Omega)$ is a Hilbert space with inner product given by

$$(u, v)_{H^1} = \int_\Omega uv\,dz + \int_\Omega (Du, Dv)_{\mathbb{R}^N}\,dz \quad \forall u, v \in H^1(\Omega).$$

By $\|\cdot\|$ we denote the corresponding norm defined by

$$\|u\| = \left(\|u\|_2^2 + \|Du\|_2^2\right)^{\frac{1}{2}} \quad \forall u \in H^1(\Omega).$$

The Banach space $C^1(\overline{\Omega})$ is an ordered Banach space with positive (order) cone

$$C_+ = \{u \in C^1(\overline{\Omega}) :\ u(z) \geqslant 0 \text{ for all } z \in \overline{\Omega}\}.$$

This cone has a nonempty interior given by

$$D_+ = \{u \in C_+ : u(z) > 0 \text{ for all } z \in \overline{\Omega}\}.$$

On $\partial\Omega$ we consider the $(N-1)$-dimensional Hausdorff (surface) measure $\sigma(\cdot)$. Using this measure, we can define in the usual way the "boundary" Lebesgue spaces $L^q(\partial\Omega)$ $(1 \leqslant q \leqslant +\infty)$. We know that there is a unique continuous linear map $\gamma_0 \colon H^1(\Omega) \longrightarrow L^2(\partial\Omega)$, known as the "trace map", such that

$$\gamma_0(u) = u|_{\partial\Omega} \quad \forall u \in H^1(\Omega) \cap C(\overline{\Omega}).$$

So, the trace map assigns boundary values to all Sobolev functions. This map is compact into $L^q(\partial\Omega)$ for all $q \in [1, \frac{2(N-1)}{N-2})$ if $N \geqslant 3$ and into $L^q(\partial\Omega)$ for all $q \geqslant 1$ if $N = 1, 2$. In addition, we have

$$\ker \gamma_0 = H_0^1(\Omega) \quad \text{and} \quad \operatorname{im} \gamma_0 = H^{\frac{1}{2},2}(\partial\Omega).$$

In what follows, for notational economy we drop the use of γ_0. All restrictions of Sobolev functions on $\partial\Omega$ are understand in the sense of traces.

Suppose that

$$\xi \in L^s(\Omega), \quad s > N$$

and

$$\beta \in W^{1,\infty}(\partial\Omega) \quad \text{with } \beta(z) \geqslant 0 \quad \forall z \in \partial\Omega.$$

Consider the following linear eigenvalue problem:

$$\begin{cases} -\Delta u(z) + \xi(z)u(z) = \widehat{\lambda} u(z) & \text{in } \Omega, \\ \frac{\partial u}{\partial n} + \beta(z)u = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.2}$$

Consider the C^1-functional $\gamma \colon H^1(\Omega) \longrightarrow \mathbb{R}$ defined by

$$\gamma(u) = \|Du\|_2^2 + \int_\Omega \xi(z)u^2\,dz + \int_{\partial\Omega} \beta(z)u^2\,d\sigma \quad \forall u \in H^1(\Omega).$$

From D'Aguì-Marano-Papageorgiou [1], we know that there exists $\mu > 0$ such that

$$\gamma(u) + \mu\|u\|_2^2 \ \geqslant\ c_0\|u\|^2 \quad \forall u \in H^1(\Omega), \tag{2.3}$$

for some $c_0 > 0$. Using (2.3) and the spectral theorem for compact self-adjoint operators on a Hilbert space, we can show that the spectrum of (2.2) consists of a strictly increasing sequence $\{\widehat{\lambda}_k\}_{k\geqslant 1}$ of eigenvalues such that $\widehat{\lambda}_k \longrightarrow +\infty$. By $E(\widehat{\lambda}_k)$, $k \in \mathbb{N}$, we denote the corresponding eigenspace. We have

- $\widehat{\lambda}_1$ is simple (that is, $\dim E(\widehat{\lambda}_1) = 1$) and

$$\widehat{\lambda}_1 \ = \ \inf\left\{\frac{\gamma(u)}{\|u\|^2} :\ u \in H^1(\Omega),\ u \neq 0\right\}; \tag{2.4}$$

- for every $m \in \mathbb{N}$, $m \geqslant 2$, we have

$$\begin{aligned}\widehat{\lambda}_m \ &= \ \inf\left\{\frac{\gamma(u)}{\|u\|^2} :\ u \in \overline{\bigoplus_{k\geqslant m} E(\widehat{\lambda}_k)},\ u \neq 0\right\} \\ &= \ \sup\left\{\frac{\gamma(u)}{\|u\|^2} :\ u \in \bigoplus_{k=1}^{m} E(\widehat{\lambda}_k),\ u \neq 0\right\};\end{aligned} \tag{2.5}$$

- for each $k \in \mathbb{N}$, $E(\widehat{\lambda}_k)$ is finite dimensional, $E(\widehat{\lambda}_k) \subseteq C^1(\overline{\Omega})$ and it has the "unique continuation property" (UCP for short), which says that if $u \in E(\widehat{\lambda}_k)$ vanishes on a set of positive measure in Ω, then $u \equiv 0$.

The above properties imply that the elements of $E(\widehat{\lambda}_1)$ do not change sign, that is,

$$E(\widehat{\lambda}_1) \ \subseteq\ C_+ \cup (-C_+).$$

In fact, if in addition we assume that $\xi^+ \in L^\infty$, then

$$E(\widehat{\lambda}_1) \setminus \{0\} \ \subseteq\ D_+ \cup (-D_+).$$

We set

$$m_+ \ = \ \min\{k \in \mathbb{N} :\ \widehat{\lambda}_k > 0\} \quad \text{and} \quad m_- \ = \ \max\{k \in \mathbb{N} :\ \widehat{\lambda}_k < 0\}.$$

Also, by $|\cdot|_N$ we denote the Lebesgue measure on $\mathbb{R}^N$. Let

$$2^* = \begin{cases} \frac{2N}{N-2} & \text{if} \quad N \geqslant 3, \\ +\infty & \text{if} \quad N = 1, 2 \end{cases}$$

(the critical Sobolev exponent) and if $\varphi \in C^1(X;\mathbb{R})$, then

$$K_\varphi = \{u \in X :\ \varphi'(u) = 0\}$$

(the critical set of φ).

By $A \in \mathcal{L}(H^1(\Omega), H^1(\Omega)^*)$ we denote the operator defined by

$$\langle A(u), h\rangle \ = \ \int_\Omega (Du, Dh)_{\mathbb{R}^N}\, dz \quad \forall u, h \in H^1(\Omega).$$

Moreover, for $q \in (1, +\infty)$, by $q' \in (1, \infty)$ we denote the conjugate exponent of q, that is,

$$\frac{1}{q} + \frac{1}{q'} \ = \ 1.$$

3 Existence Theorem

In this section we prove an existence theorem for problem (1.1). We impose the following conditions on the data of (1.1).

$\underline{H(\xi)}$: $\xi \in L^s(\Omega)$, $s > N$.

$\underline{H(\beta)}$: $\beta \in W^{1,\infty}(\partial\Omega)$ with $\beta(z) \geqslant 0$ for all $z \in \partial\Omega$.

Remark 3.1. When $\beta \equiv 0$, we have the usual Neumann problem.

$\underline{H_1}$: $f\colon \Omega \times \mathbb{R} \longrightarrow \mathbb{R}$ is a Carathéodory function such that

(i) for every $\varrho > 0$, there exists $a_\varrho \in L^\infty(\Omega)$ such that

$$|f(z,\zeta)| \;\leqslant\; a_\varrho(z) \quad \text{for a.a. } z \in \Omega, \text{ all } |\zeta| \leqslant \varrho$$

and

$$\lim_{\zeta \to \pm\infty} \frac{f(z,\zeta)}{|\zeta|^{2^*-2}\zeta} \;=\; 0 \quad \text{uniformly for a.a. } z \in \Omega;$$

(ii) if

$$F(z,\zeta) \;=\; \int_0^\zeta f(z,s)\,ds$$

and

$$e(z,\zeta) \;=\; f(z,\zeta)\zeta - 2F(z,\zeta),$$

then

$$\lim_{\zeta \to \pm\infty} \frac{F(z,\zeta)}{\zeta^2} \;=\; +\infty \quad \text{uniformly for a.a. } z \in \Omega$$

and there exist $d \in L^1(\Omega)$ and $k \in \mathbb{N}$ such that

$$e(z,s\zeta) \;\leqslant\; ke(z,\zeta) + d(z) \quad \text{for a.a. } z \in \Omega, \text{ all } \zeta \in \mathbb{R}, \; s \in [0,1].$$

(iii) we have

$$\lim_{\zeta \to 0} \frac{f(z,\zeta)}{\zeta} \;=\; 0 \quad \text{uniformly for a.a. } z \in \Omega$$

and there exists $\delta > 0$ such that

$$F(z,\zeta) \;\leqslant\; 0 \quad \text{for a.a. } z \in \Omega, \text{ all } |\zeta| \leqslant \delta.$$

Remark 3.2. Hypothesis $H_1(i)$ is more general than the usual subcritical polynomial growth condition which says that

$$|f(z,\zeta)| \;\leqslant\; a(z)(1 + |\zeta|^{r-1}) \quad \text{for a.a. } z \in \Omega, \text{ all } \zeta \in \mathbb{R},$$

with $a \in L^\infty(\Omega)$, $2 < r < 2^*$. Hypothesis $H_1(i)$ implies that given $\varepsilon > 0$, we can find $a_\varepsilon \in L^\infty(\Omega)$ such that

$$|f(z,\zeta)| \leqslant a_\varepsilon(z) + \varepsilon|\zeta|^{2^*-1} \quad \text{for a.a. } z \in \Omega, \text{ all } \zeta \in \mathbb{R}. \tag{3.1}$$

So, $f(z,\cdot)$ exhibits almost critical growth. The lack of compactness in the embedding of $H^1(\Omega)$ into $L^{2^*}(\Omega)$ is a source of difficulties in the study of problem (1.1). We overcome these difficulties without any use of the concentration-compactness principle. Instead our method of proof uses Vitali's theorem (the extended dominated convergence theorem; see Gasiński-Papageorgiou [4, p. 901]).

Hypothesis $H_1(ii)$ is the superlinearity condition on $f(z,\cdot)$. It implies that

$$\lim_{\zeta\to\pm\infty} \frac{f(z,\zeta)}{\zeta} = +\infty \quad \text{uniformly for a.a. } z \in \Omega,$$

that is, $f(z,\cdot)$ is superlinear. This superlinearity of $f(z,\cdot)$ is not expressed using the usual in such cases Ambrosetti-Rabinowitz condition. Recall that the Ambrosetti-Rabinowitz condition says that there exist $r > 2$ and $M > 0$ such that

$$0 < rF(z,\zeta) \leqslant f(z,\zeta)\zeta \quad \text{for a.a. } z \in \Omega, \text{ all } |\zeta| \geqslant M, \tag{3.2}$$

and

$$0 < \operatorname*{ess\,inf}_{\Omega} F(\cdot,\pm M). \tag{3.3}$$

Integrating (3.2) and using (3.3), we obtain the weaker condition

$$c_1|\zeta|^r \leqslant F(z,\zeta) \quad \text{for a.a. } z \in \Omega, \text{ all } |\zeta| \geqslant M, \tag{3.4}$$

with $c_1 > 0$. From (3.4) and (3.2), it follows that the Ambrosetti-Rabinowitz condition implies that $f(z,\cdot)$ has at least $(r-1)$-polynomial growth. This excludes from consideration superlinear functions with slower growth (see the examples below). Our hypothesis $H_1(ii)$ is a more general version of a condition used by Li-Yang [11]. It is satisfied if there exists $M > 0$ such that for almost all $z \in \Omega$:

$$\zeta \longmapsto \frac{f(z,\zeta)}{\zeta} \quad \text{is nondecreasing on } [M,+\infty),$$

$$\zeta \longmapsto \frac{f(z,\zeta)}{\zeta} \quad \text{is nonincreasing on } (-\infty,M].$$

Hypothesis $H_1(iii)$ implies that $f(z,\cdot)$ is strictly sublinear near zero. Also, from that hypothesis we have that

$$f(z,0) = 0 \quad \text{for a.a. } z \in \Omega.$$

Therefore the trivial function $u \equiv 0$ is always a solution of problem (1.1). Our aim is to produce nonzero solutions.

Example 3.3. *The following functions satisfy hypotheses H_1. For the sake of simplicity we drop the z-dependence.*

$$f_1(\zeta) = \begin{cases} |\zeta|^{r-2}\zeta - |\zeta|^{\tau-2}\zeta & \text{if} \quad |\zeta| \leqslant 1, \\ \zeta \ln \zeta & \text{if} \quad |\zeta| > 1, \end{cases}$$

$$f_2(\zeta) = \begin{cases} |\zeta|^{2^*-2}\zeta(\ln(1+|\zeta|) - \frac{1}{2^*}\frac{|\zeta|}{1+|\zeta|}) + c & \text{if} \quad \zeta < -1, \\ |\zeta|^{r-2}\zeta - |\zeta|^{\tau-2}\zeta & \text{if} \quad -1 \leqslant \zeta \leqslant 1, \\ |\zeta|^{2^*-2}\zeta(\ln(1+|\zeta|) - \frac{1}{2^*}\frac{|\zeta|}{1+|\zeta|}) - c & \text{if} \quad \zeta > 1, \end{cases}$$

with $2 < \tau < r < +\infty$ and $c = \frac{1}{\ln 2} - \frac{1}{2^* 2}$.

Note that f_1 fails to satisfy the Ambrosetti-Rabinowitz condition, while f_2 does not have a subcritical polynomial growth.

Consider the energy (Euler) functional for problem (1.1), $\varphi\colon H^1(\Omega) \longrightarrow \mathbb{R}$ defined by

$$\varphi(u) = \frac{1}{2}\gamma(u) - \int_\Omega F(z,u)\,dz \quad \forall u \in H^1(\Omega).$$

Evidently $\varphi \in C^1(H^1(\Omega))$.

Proposition 3.4. *If hypotheses $H(\xi)$, $H(\beta)$, H_1 hold, then the energy functional φ satisfies the $(C)^*$-condition.*

Proof. We consider a sequence $\{u_n\}_{n\geqslant 1} \subseteq H^1(\Omega)$ such that

$$\varphi(u_n) \leq M_1 \quad \forall n \in \mathbb{N}, \tag{3.5}$$

for some $M_1 > 0$ and

$$(1+\|u_n\|)\varphi'(u_n) \longrightarrow 0 \quad \text{in } H^1(\Omega)^*. \tag{3.6}$$

From (3.6) we have

$$\begin{aligned} &\left|\langle A(u_n), h\rangle + \int_\Omega \xi(z)u_n h\,dz + \int_{\partial\Omega} \beta(z)u_n h\,d\sigma - \int_\Omega f(z,u_n)h\,dz\right| \\ \leqslant\ & \frac{\varepsilon_n\|h\|}{1+\|u_n\|} \quad \forall h \in H^1(\Omega), \end{aligned} \tag{3.7}$$

with $\varepsilon_n \searrow 0$. In (3.7) we choose $h = u_n \in H^1(\Omega)$. We obtain

$$-\gamma(u_n) + \int_\Omega f(z,u_n)u_n\,dz \leqslant \varepsilon_n \quad \forall n \in \mathbb{N}. \tag{3.8}$$

From (3.5) we have

$$\gamma(u_n) - \int_\Omega 2F(z,u_n)\,dz \leqslant 2M_1 \quad \forall n \in \mathbb{N}. \tag{3.9}$$

We add (3.8) and (3.9) and obtain

$$\int_\Omega e(z,u_n)\,dz \leqslant M_2 \quad \forall n \in \mathbb{N}, \tag{3.10}$$

for some $M_2 > 0$.

Claim: The sequence $\{u_n\}_{n \geqslant 1} \subseteq H^1(\Omega)$ is bounded.

We argue indirectly. So, suppose that the Claim is not true. Passing to a suitable subsequence if necessary, we may assume that

$$\|u_n\| \longrightarrow +\infty \quad \text{as } n \to +\infty. \tag{3.11}$$

We set $y_n = \frac{u_n}{\|u_n\|}$, $n \in \mathbb{N}$. Then $\|y_n\| = 1$ for all $n \in \mathbb{N}$ and so, passing to a next subsequence if necessary, we may assume that

$$y_n \xrightarrow{w} y \quad \text{in } H^1(\Omega) \quad \text{and} \quad y_n \longrightarrow y \quad \text{in } L^{\frac{2s}{s-1}}(\Omega) \text{ and in } L^2(\partial\Omega). \tag{3.12}$$

First we assume that $y \not\equiv 0$. Let $\Omega_0 = \{z \in \Omega : \; y(z) \neq 0\}$. Then $|\Omega_0|_N > 0$ and

$$|u_n(z)| \longrightarrow +\infty \quad \text{for a.a. } z \in \Omega_0.$$

Hypothesis $H_1(ii)$ implies that

$$\frac{F(z, u_n(z))}{\|u_n\|^2} = \frac{F(z, u_n(z))}{u_n(z)^2} y_n(z)^2 \longrightarrow +\infty \quad \text{for a.a. } z \in \Omega_0.$$

Fatou's lemma implies that

$$\int_{\Omega_0} \frac{F(z, u_n(z))}{\|u_n\|^2} \, dz \longrightarrow +\infty \tag{3.13}$$

(see hypothesis $H_1(ii)$). Hypotheses $H_1(i)$ and (ii) imply that we can find $c_2 > 0$ such that

$$-c_2 \leqslant F(z, \zeta) \quad \text{for a.a. } z \in \Omega, \; \text{all } \zeta \in \mathbb{R}.$$

Therefore

$$\begin{aligned} \int_\Omega \frac{F(z, u_n(z))}{\|u_n\|^2} \, dz &= \int_{\Omega_0} \frac{F(z, u_n(z))}{\|u_n\|^2} \, dz + \int_{\Omega \setminus \Omega_0} \frac{F(z, u_n(z))}{\|u_n\|^2} \, dz \\ &\geqslant \int_{\Omega_0} \frac{F(z, u_n(z))}{\|u_n\|^2} \, dz - \frac{c_2}{\|u_n\|^2} |\Omega|_N, \end{aligned}$$

so

$$\int_\Omega \frac{F(z, u_n(z))}{\|u_n\|^2} \, dz \longrightarrow +\infty \tag{3.14}$$

(see (3.13) and (3.11)). From hypothesis $H_1(ii)$, we have

$$2kF(z, \zeta) \leqslant k f(z, \zeta)\zeta + d(z) \quad \text{for a.a. } z \in \Omega, \; \text{all } \zeta \in \mathbb{R}. \tag{3.15}$$

From (3.7) with $h = u_n \in H^1(\Omega)$, we have

$$\int_\Omega f(z, u_n) u_n \, dz \leqslant M_3 + \gamma(u_n) \quad \forall n \in \mathbb{N},$$

for some $M_3 > 0$, so

$$2k \int_\Omega F(z, u_n)\, dz \ \leqslant\ M_4 + k\gamma(u_n) \quad \forall n \in \mathbb{N},$$

for some $M_4 > 0$ (see (3.15)), thus

$$2k \int_\Omega \frac{F(z, u_n)}{\|u_n\|^2}\, dz \ \leqslant\ \frac{M_4}{\|u_n\|^2} + k\gamma(y_n) \quad \forall n \in \mathbb{N},$$

hence

$$\int_\Omega \frac{F(z, u_n(z))}{\|u_n\|^2}\, dz \ \leqslant\ M_5 \quad \forall n \in \mathbb{N}, \tag{3.16}$$

for some $M_5 > 0$ (recall that $\|y_n\| = 1$ for all $n \in \mathbb{N}$). Comparing (3.14) and (3.16), we reach a contradiction.

Now suppose that $y \equiv 0$. Let $\vartheta > 0$ and set $v_n = (2\vartheta)^{\frac{1}{2}} y_n$ for all $n \in \mathbb{N}$. From (3.12) and since $y \equiv 0$, we have

$$v_n \ \xrightarrow{w}\ 0 \quad \text{in } H^1(\Omega) \quad \text{and} \quad v_n \ \longrightarrow\ 0 \quad \text{in } L^{\frac{2s}{s-1}} \text{ and in } L^2(\partial\Omega). \tag{3.17}$$

Let $c_3 = \sup\limits_{n \geqslant 1} \|v_n\|_{2^*}^{2^*} < +\infty$ (see (3.17)). Hypothesis $H_1(i)$ implies that given $\varepsilon > 0$ we can find $c_\varepsilon > 0$ such that

$$|F(z, \zeta)| \ \leqslant\ \frac{\varepsilon}{2c_\varepsilon} |\zeta|^{2^*} + c_\varepsilon \quad \text{for a.a. } z \in \Omega, \text{ all } \zeta \in \mathbb{R}. \tag{3.18}$$

Suppose that $E \subseteq \Omega$ is a measurable set and $|E|_N \leqslant \frac{\varepsilon}{2c_\varepsilon}$. Then

$$\left| \int_E F(z, v_n)\, dz \right| \ \leqslant\ \int_E |F(z, v_n)|\, dz \ \leqslant\ \frac{\varepsilon}{2c_\varepsilon} \|v_n\|_{2^*}^{2^*} + c_\varepsilon |\Omega|_N \ \leqslant\ \varepsilon$$

(see (3.18)), so

$$\{F(\cdot, v_n(\cdot))\}_{n \geqslant 1} \subseteq L^1(\Omega) \text{ is uniformly integrable.} \tag{3.19}$$

Also, we have

$$F(z, v_n(z)) \ \longrightarrow\ 0 \quad \text{for a.a. } z \in \Omega. \tag{3.20}$$

From (3.19), (3.20) and Vitali's theorem (the extended dominated convergence theorem; see Gasiński-Papageorgiou [4, p. 901]), we have

$$\int_\Omega F(z, v_n)\, dz \ \longrightarrow\ 0. \tag{3.21}$$

From (3.11) we see that we can find $n_0 \in \mathbb{N}$ such that

$$0 \ <\ (2\vartheta)^{\frac{1}{2}} \frac{1}{\|u_n\|} \ \leqslant\ 1 \quad \forall n \geqslant n_0. \tag{3.22}$$

We choose $t_n \in [0,1]$ such that

$$\varphi(t_n u_n) = \max\{\varphi(tu_n) : 0 \leqslant t \leqslant 1\} \quad \forall n \in \mathbb{N}. \tag{3.23}$$

From (3.22) and (3.23), we have

$$\begin{aligned} \varphi(t_n u_n) &\geqslant \varphi(v_n) = \vartheta\gamma(y_n) - \int_\Omega F(z, v_n)\,dz \\ &= \vartheta\big(\gamma(y_n) + \mu\|y_n\|_2^2\big) - \int_\Omega \big(F(z, v_n) + \vartheta\mu y_n^2\big)\,dz \\ &\geqslant \vartheta c_0 - \int_\Omega \big(F(z, v_n) + \vartheta\mu y_n^2\big)\,dz \quad \forall n \geqslant n_0 \end{aligned} \tag{3.24}$$

(see (2.3) and recall that $\|y\| = 1$ for all $n \in \mathbb{N}$). Recall that $y = 0$. So, from (3.12) (note that $\frac{2s}{s-1} > 2$) and (3.21) we have

$$\int_\Omega \big(F(z, v_n) + \vartheta\mu y_n^2\big)\,dz \longrightarrow 0.$$

So, we can find $n_1 \in \mathbb{N}$, $n_1 \geqslant n_0$, such that

$$\int_\Omega \big(F(z, v_n) + \vartheta\mu y_n^2\big)\,dz \leqslant \frac{1}{2}\vartheta c_0 \quad \forall n \geqslant n_1. \tag{3.25}$$

Returning to (3.24) and using (3.25), we obtain

$$\varphi(t_n u_n) \geqslant \frac{1}{2}\vartheta c_0 \quad \forall n \geqslant n_1.$$

But $\vartheta > 0$ is arbitrary. So, we infer that

$$\varphi(t_n u_n) \longrightarrow +\infty. \tag{3.26}$$

We have

$$\varphi(u_n) \leqslant M_1 \quad \forall n \in \mathbb{N} \quad \text{and} \quad \varphi(0) = 0 \tag{3.27}$$

(see (3.5)). From (3.23), (3.26) and (3.27), we see that we can find $n_2 \in \mathbb{N}$ such that

$$t_n \in (0,1) \quad \forall n \geqslant n_2. \tag{3.28}$$

So, we have

$$\frac{d}{dt}\varphi(tu_n)\big|_{t=t_n} = 0 \quad \forall n \geqslant n_2$$

(see (3.23)), so

$$\langle \varphi'(t_n u_n), t_n u_n \rangle = 0 \quad \forall n \geqslant n_2$$

(using the chain rule and (3.28)), thus

$$\gamma(t_n u_n) = \int_\Omega f(z, t_n u_n)(t_n u_n)\,dz \quad \forall n \geqslant n_2. \tag{3.29}$$

Then (3.28) and hypothesis $H_1(ii)$ imply that

$$\int_\Omega e(z, t_n u_n)\, dz \ \leqslant \ k \int_\Omega e(z, u_n)\, dz + \|d\|_1 \quad \forall n \geqslant n_2,$$

so

$$\begin{aligned} & \int_\Omega f(z, t_n u_n)(t_n u_n)\, dz \ \leqslant \ k \int_\Omega e(z, u_n)\, dz + \int_\Omega 2F(z, t_n u_n)\, dz + \|d\|_1 \\ \leqslant \ & M_6 + \int_\Omega 2F(z, t_n u_n)\, dz \quad \forall n \geqslant n_2, \end{aligned} \tag{3.30}$$

for some $M_6 > 0$ (see (3.10)). We return to (3.29) and use (3.30). Then

$$2\varphi(t_n u_n) \ \leqslant \ M_6 \quad \forall n \geqslant n_2. \tag{3.31}$$

Comparing (3.26) and (3.31) we have a contradiction. This proves the Claim.

On the account of the Claim, passing to a subsequence if necessary, we may assume that

$$u_n \xrightarrow{w} u \quad \text{in } H^1(\Omega) \quad \text{and} \quad u_n \longrightarrow u \quad \text{in } L^{\frac{2s}{s-1}}(\Omega) \quad \text{and in } L^2(\partial\Omega). \tag{3.32}$$

Let $c_4 = \sup_{n \geqslant 1} \|u_n\|_{2^*} < +\infty$ (see (3.32)). Hypothesis $H_1(i)$ implies that given $\varepsilon > 0$ we can find $\widehat{c}_\varepsilon > 0$ such that

$$|f(z,\zeta)| \ \leqslant \ \frac{\varepsilon}{2c_4^{2^*}} |\zeta|^{2^*-1} + \widehat{c}_\varepsilon \text{ for a.a. } z \in \Omega, \text{ all } \zeta \in \mathbb{R}. \tag{3.33}$$

For a measurable set $E \subseteq \Omega$, we have

$$\begin{aligned} & \left| \int_E f(z, u_n)(u_n - u)\, dz \right| \ \leqslant \ \int_E |f(z, u_n)|\, |u_n - u|\, dz \\ \leqslant \ & \frac{\varepsilon}{2c_4} \int_E |u_n|^{2^*-1} |u_n - u|\, dz + \widehat{c}_4 \int_\Omega |u_n - u|\, dz \quad \forall n \in \mathbb{N} \end{aligned} \tag{3.34}$$

(see (3.33)). Note that

$$|u_n|^{2^*-1} \ \in \ L^{(2^*)'}(\Omega) \quad \text{and} \quad |u_n - u| \ \in \ L^{2^*}(\Omega)$$

(recall that $2^* - 1 = \frac{2^*}{(2^*)'}$). Using Hölder inequality, we have

$$\begin{aligned} & \frac{\varepsilon}{2c_4^{2^*}} \int_E |u_n|^{2^*-1} |u_n - u|\, dz \\ \leqslant \ & \frac{\varepsilon}{2c_4^{2^*}} \|u\|_{2^*}^{2^*-1} \|u_n - u\|_{2^*} \ \leqslant \ \frac{\varepsilon}{2} \quad \forall n \in \mathbb{N}. \end{aligned} \tag{3.35}$$

Similarly, we have

$$\widehat{c}_\varepsilon \int_E |u_n - u|\, dz \ \leqslant \ \widehat{c}_\varepsilon |E|_N^{\frac{1}{(2^*)'}} \|u_n - u\|_{2^*} \ \leqslant \ 2\widehat{c}_\varepsilon c_4 |E|_N^{\frac{1}{(2^*)'}}. \tag{3.36}$$

We assume that

$$|E|_N \leqslant \left(\frac{\varepsilon}{4}\frac{1}{\widehat{c}_\varepsilon c_4}\right)^{(2^*)'}. \tag{3.37}$$

Using (3.37) in (3.36), we see that

$$\widehat{c}_\varepsilon \int_E |u_n - u|\,dz \leqslant \frac{\varepsilon}{2} \quad \forall n \in \mathbb{N}. \tag{3.38}$$

From (3.35) and (3.38), we see that given $\varepsilon > 0$, we can find $\widehat{\delta} = (\frac{\varepsilon}{4}\frac{1}{\widehat{c}_\varepsilon c_4})^{(2^*)'}$ such that

$$\text{if } |E|_N \leqslant \widehat{\delta}, \text{ then } \sup_{n\geqslant 1}\int_E |f(z,u_n)|\,|u_n - u|\,dz \leqslant \varepsilon,$$

so

$$\text{the sequence } \{f(\cdot,u_n(\cdot))(u_n-u)(\cdot)\}_{n\geqslant 1} \text{ is uniformly integrable.}$$

For at least a subsequence, we have

$$f(z,u_n(z))(u_n-u)(z) \longrightarrow 0 \quad \text{for a.a. } z \in \Omega.$$

Therefore Vitali's theorem implies that

$$\int_\Omega f(z,u_n)(u_n-u)\,dz \longrightarrow 0. \tag{3.39}$$

In (3.7) we choose $h = u_n - u \in H^1(\Omega)$, pass to the limit as $n \to +\infty$ and use (3.32) and (3.39). Then

$$\lim_{n\to+\infty}\langle A(u_n), u_n - u\rangle = 0,$$

so

$$\|Du_n\|_2 \longrightarrow \|Du\|_2,$$

thus

$$u_n \longrightarrow u \quad \text{in } H^1(\Omega)$$

(by the Kadec-Klee property; see Gasiński-Papageorgiou [4, p. 911]) and hence φ satisfies the $(C)^*$-condition. $\square$

We consider the following orthogonal direct sum decomposition

$$H^1(\Omega) = H_- \oplus V, \tag{3.40}$$

with

$$H_- = \bigoplus_{i=1}^{m_-} E(\widehat{\lambda}_i) \quad \text{and} \quad V = H_-^\perp = E(0) \oplus H_+, \tag{3.41}$$

where $H_+ = \overline{\bigoplus_{i\geqslant m_+} E(\widehat{\lambda}_i)}$.

Proposition 3.5. *If hypotheses $H(\xi)$, $H(\beta)$ and H_1 hold, then φ has at $u = 0$ a local linking with respect to the decomposition (3.40).*

Proof. From (3.41), every $v \in V$ admits a unique sum decomposition

$$v \;=\; v^0 + \widehat{v}, \quad \text{with } v^0 \in E(0),\; \widehat{v} \in H_+.$$

The eigenspace $E(0)$ is finite dimensional. So, all norms on $E(0)$ are equivalent and we can find $c_5 > 0$ such that

$$\|v^0\|_\infty \;\leqslant\; c_5\|v^0\| \quad \forall v^0 \in E(0). \tag{3.42}$$

Let $\delta > 0$ be as postulated by hypothesis $H_1(iii)$. We introduce the following measurable subsets of Ω

$$\Omega_1 \;=\; \Big\{z \in \Omega : |\widehat{v}(z)| \leqslant \frac{\delta}{2}\Big\} \quad \text{and} \quad \Omega_2 \;=\; \Omega \setminus \Omega_1. \tag{3.43}$$

Suppose that $z \in \Omega_1$. We have

$$|v(z)| \;\leqslant\; |v^0(z)| + |\widehat{v}(z)| \;\leqslant\; c_5\|v^0\| + \frac{\delta}{2} \tag{3.44}$$

(see (3.42), (3.43)).

So, if $\varrho_1 = \frac{\delta}{2c_5}$ and $v \in V$ satisfies $\|v\| \leqslant \varrho_1$, then from (3.44) we have

$$|v(z)| \;\leqslant\; \delta \quad \forall z \in \Omega_1$$

(recall that $\|v^0\| \leqslant \|v\|$), so

$$\int_{\Omega_1} F(z,v)\,dz \;\leqslant\; 0 \quad \forall v \in V,\; \|v\| \leqslant \varrho_1 \tag{3.45}$$

(see hypothesis $H_1(iii)$). Hypotheses $H_1(i),(iii)$ imply that given $\varepsilon > 0$, we can find $c_6 = c_6(\varepsilon) > 0$ such that

$$F(z,\zeta) \;\leqslant\; \varepsilon\zeta^2 + c_6|\zeta|^{2^*} \quad \text{for a.a. } z \in \Omega,\; \text{all } \zeta \in \mathbb{R}. \tag{3.46}$$

Suppose that $v \in V$ satisfies $\|v\| \leqslant \varrho_1$ and $z \in \Omega_2$. Then

$$|v(z)| \;\leqslant\; |v^0(z)| + |\widehat{v}(z)| \;\leqslant\; 2|\widehat{v}(z)| \tag{3.47}$$

(see (3.42), (3.43)). From (3.46) and (3.47), we have

$$\int_{\Omega_2} F(z,v(z))\,dz \;\leqslant\; 4\varepsilon\|\widehat{v}\|_2^2 + c_7\|\widehat{v}\|_{2^*}^{2^*} \quad \forall v \in V,\; \|v\| \leqslant \varrho_1, \tag{3.48}$$

for some $c_7 > 0$. Exploiting the orthogonality of the component spaces in (3.41), we have

$$\varphi(v) \;=\; \frac{1}{2}\gamma(\widehat{v}) - \int_\Omega F(z,v)\,dz \;\geqslant\; (c_8 - 4\varepsilon)\|\widehat{v}\|^2 - c_9\|\widehat{v}\|^{2^*} \quad \forall v \in V,\; \|v\| \leqslant \varrho_1,$$

for some $c_8, c_9 > 0$ (recall that $v^0 \in E(0)$, $\widehat{v} \in H_+$ and see (3.45), (3.48)).

Choosing $\varepsilon \in (0, \frac{c_8}{4})$, we see that

$$\varphi(v) \geqslant c_{10}\|\widehat{v}\|^2 - c_9\|\widehat{v}\|^{2^*} \quad \forall v \in V,\ \|v\| \leqslant \varrho_1, \tag{3.49}$$

for some $c_{10} > 0$. Since $2^* > 2$, from (3.49) and by choosing $\varrho_2 \in (0, \min\{1, \varrho_1\})$ small, we have

$$\varphi(v) > 0 \quad \forall v \in V,\ 0 < \|v\| \leqslant \varrho_2. \tag{3.50}$$

Hypotheses $H_1(i), (iii)$ imply that given $\varepsilon > 0$, we can find $c_{11} = c_{11}(\varepsilon) > 0$ such that

$$F(z,\zeta) \geqslant -\frac{\varepsilon}{2}\zeta^2 - c_{11}|\zeta|^{2^*} \quad \text{for a.a. } z \in \Omega,\ \text{all } \zeta \in \mathbb{R}. \tag{3.51}$$

Then for $\overline{u} \in H_-$, we have

$$\varphi(\overline{u}) = \frac{1}{2}\gamma(\overline{u}) - \int_\Omega F(z,\overline{u})\,dz \leqslant -c_{12}\|\overline{u}\|^2 + c_{13}\|u\|^{2^*},$$

for some $c_{12}, c_{13} > 0$ (choosing $\varepsilon > 0$ small enough). Since $2^* > 2$, choosing $\varphi_3 \in (0,1)$ small, we have

$$\varphi(\overline{u}) \leqslant 0 \quad \forall \overline{u} \in H_-,\ \|\overline{u}\| \leqslant \varrho_3. \tag{3.52}$$

From (3.50) and (3.52), it follows that φ has at $u = 0$ a local linking with respect to the decomposition (3.40). □

Proposition 3.6. *If hypotheses $H(\xi)$, $H(\beta)$ and H_1 hold and $E \subseteq V$ (see (3.41)) is a finite dimensional subspace, then $\varphi(u) \longrightarrow -\infty$ as $\|u\| \to +\infty$ with $u \in H_- \oplus E$.*

Proof. Hypotheses $H_1(i), (ii)$ imply that given any $\eta > 0$, we can find $c_{14} = c_{14}(\eta) > 0$ such that

$$F(z,\zeta) \geqslant \eta\zeta^2 - c_{14} \quad \text{for a.a. } z \in \Omega,\ \text{all } \zeta \in \mathbb{R}. \tag{3.53}$$

The space $H_- \oplus E$ is finite dimensional and so all norms are equivalent. Then for $u \in H_- \oplus E$ we have

$$\varphi(u) = \frac{1}{2}\gamma(u) - \int_\Omega F(z,u)\,dz \leqslant (c_{15} - \eta)\|u\|^2 + c_{16},$$

for some $c_{15}, c_{16} > 0$ (see (3.53)). Choosing $\eta > c_{15}$, we see that

$$\varphi(u) \longrightarrow -\infty \quad \text{as } \|u\| \to +\infty, \quad \text{with } u \in H_- \oplus E.$$

□

Now we are ready for the existence theorem.

Theorem 3.7. *If hypotheses $H(\xi)$, $H(\beta)$ and H_1 hold, then problem (1.1) admits a nontrivial solution $u_0 \in C^1(\overline{\Omega})$.*

Proof. Evidently φ maps bounded sets to bounded sets. This fact and Propositions 3.4, 3.5 and 3.6, permit the use of Theorem 2.2 and find $u_0 \in H^1(\Omega)$ such that

$$u_0 \in K_\varphi \setminus \{0\}.$$

We have

$$\langle A(u_0), h\rangle + \int_\Omega \xi(z) u_0 h\, dz + \int_{\partial\Omega} \beta(z) u_0 h\, d\sigma = \int_\Omega f(z, u_0) h\, dz \quad \forall h \in H^1(\Omega),$$

so

$$\begin{cases} -\Delta u_0(z) + \xi(z) u_0(z) = f(z, u_0(z)) & \text{in } \Omega, \\ \frac{\partial u_0}{\partial n} + \beta(z) u_0 = 0 & \text{on } \partial\Omega \end{cases} \tag{3.54}$$

(see Papageorgiou-Rădulescu [15]). Let

$$\widehat{a}(z) = \begin{cases} 0 & \text{if } |u_0(z)| \leqslant 1, \\ \frac{f(z,u_0(z))}{u_0(z)} & \text{if } |u_0(z)| > 1 \end{cases} \tag{3.55}$$

$$\widehat{b}(z) = \begin{cases} f(z, u_0(z)) & \text{if } |u_0(z)| \leqslant 1, \\ 0 & \text{if } |u_0(z)| > 1. \end{cases} \tag{3.56}$$

Hypotheses $H_1(i)$ and (iii) imply that given $\varepsilon > 0$, we can find $c_{17} = c_{17} > 0$ such that

$$|f(z,\zeta)| \leqslant \varepsilon |\zeta|^{2^*-1} + c_{17}|\zeta| \quad \text{for a.a. } z \in \Omega, \text{ all } \zeta \in \mathbb{R}. \tag{3.57}$$

Using (3.57) and the Sobolev embedding theorem, we see that

$$\widehat{a} \in L^{\frac{N}{2}}(\Omega).$$

Also, from (3.35) and hypothesis $H_1(i)$, we have

$$\widehat{b} \in L^\infty(\Omega).$$

From (3.54) we obtain

$$\begin{cases} -\Delta u_0(z) = (\widehat{a}(z) - \xi(z)) u_0(z) + \widehat{b}(z) & \text{in } \Omega, \\ \frac{\partial u_0}{\partial n} + \beta(z) u_0 = 0 & \text{on } \partial\Omega \end{cases} \tag{3.58}$$

Note that $\widehat{a} - \xi \in L^{\frac{N}{2}}(\Omega)$ (see hypothesis $H(\xi)$) and $\widehat{b} \in L^\infty(\Omega)$. So, using Lemma 5.1 of Wang [22], we have that

$$u_0 \in L^\infty(\Omega)$$

(see (3.58)). Hypotheses $H_1(i)$ and $H(\xi)$ imply that

$$f(\cdot, u_0(\cdot)) - \xi(\cdot) u_0(\cdot) \in L^s(\Omega).$$

So, Lemma 5.2 of Wang [22] (the Calderon-Zygmund estimates) implies that

$$u_0 \in W^{2,s}(\Omega),$$

thus

$$u_0 \in C^{1,\alpha}(\overline{\Omega}), \quad \text{with } \alpha = 1 - \frac{N}{s} > 0$$

(by the Sobolev embedding theorem). □

4 Infinitely Many Solutions

In this section we prove a theorem producing an unbounded sequence of distinct smooth solutions.

We introduce the following conditions on the reaction term $f(z,\zeta)$:

$\underline{H_2}$: $f\colon \Omega\times\mathbb{R} \longrightarrow \mathbb{R}$ is a Carathéodory function such that $f(z,\cdot)$ is odd for almost all $z \in \Omega$ and hypotheses $H_2(i)$ and (ii) are the same as the corresponding hypotheses $H_1(i)$ and (ii).

Remark 4.1. Note that in this case no condition near zero is imposed (see hypothesis $H_1(iii)$). Instead, we have a symmetry condition on $f(z,\cdot)$, namely we require that $f(z,\cdot)$ is odd.

Recall that

$$H^1(\Omega) = H_- \oplus E(0) \oplus H_+,$$

with

$$H_- = \bigoplus_{i=1}^{m_-} E(\widehat{\lambda}_i) \quad \text{and} \quad H_+ = \overline{\bigoplus_{i\geqslant m_+} E(\widehat{\lambda}_i)}.$$

Proposition 4.2. *If hypotheses $H(\xi)$, $H(\beta)$ and H_2 hold, then there exist $\eta, r > 0$ and a subspace $E \subseteq H_+$ such that*

$$\varphi\big|_{E\cap\partial B_\varrho} \geqslant \eta > 0.$$

Proof. Hypothesis $H_2(i)$ implies that given $\varepsilon > 0$, we can find $c_{17} = c_{17}(\varepsilon) > 0$ such that

$$F(z,\zeta) \leqslant \varepsilon|\zeta|^{2^*} + c_{17}|\zeta| \quad \text{for a.a. } z\in\Omega, \text{ all } \zeta\in\mathbb{R}. \tag{4.1}$$

For $u \in H_+$, we have

$$\begin{aligned} \varphi(u) &= \frac{1}{2}\gamma(u) - \int_\Omega F(z,u)\,dz \geqslant \frac{1}{2}\gamma(u) - \varepsilon\|u\|_{2^*}^{2^*} - c_{17}\|u\|_1 \\ &\geqslant \frac{1}{2}\gamma(u) - \varepsilon c_{18}\|u\|^{2^*} - c_{19}\frac{\|u\|}{\sqrt{\widehat{\lambda}_n}} \end{aligned}$$

$$
\begin{aligned}
&\geqslant c_{20}\|u\|^2 - \varepsilon c_{18}\|u\|^{2^*} - \frac{c_{19}}{\sqrt{\widehat{\lambda}_n}}\|u\| \\
&= \left(\frac{c_{20}}{2}\|u\|^2 - \varepsilon c_{18}\|u\|^{2^*}\right) + \left(\frac{c_{20}}{2}\|u\|^2 - \frac{c_{19}}{\sqrt{\widehat{\lambda}_n}}\|u\|\right), \qquad (4.2)
\end{aligned}
$$

for some $c_{18}, c_{19}, c_{20} > 0$ and all $n \in \mathbb{N}$, $n \geqslant m_+$ (recall that $u \in H_+$). For $\varepsilon \in (0,1)$, we can always find $\widehat{u}_0 \in H_+$, $\|\widehat{u}_0\| < 1$ such that

$$
\frac{c_{20}}{2}\|\widehat{u}_0\|^2 - \varepsilon c_{18}\|\widehat{u}_0\|^{2^*} > 0 \qquad (4.3)
$$

(recall that $2 < 2^*$). Then we choose $n \in \mathbb{N}$, $n \geqslant m_+$ such that

$$
\widehat{\lambda}_n \geqslant \left(\frac{2c_{19}}{c_{20}}\frac{1}{\|\widehat{u}_0\|}\right)^2
$$

(recall that $\widehat{\lambda}_n \to +\infty$). We consider the following orthogonal direct sum decomposition of $H^1(\Omega)$:

$$
H^1(\Omega) = Y \oplus E,
$$

with

$$
Y = \bigoplus_{i=1}^{n-1} E(\widehat{\lambda}_i) \quad \text{and} \quad E = Y^\perp = \overline{\bigoplus_{i \geqslant n} E(\widehat{\lambda}_i)}.
$$

Then for $u \in E$ with $\|u\| = \|\widehat{u}_0\| = r < 1$, we have

$$
\varphi(u) = \eta = \frac{c_{20}}{2}\|\widehat{u}_0\|^2 - \varepsilon c_{18}\|\widehat{u}_0\|^{2^*} > 0
$$

(see (4.2) and (4.3)). □

Proposition 4.3. *If hypotheses $H(\xi)$, $H(\beta)$ and H_2 hold and $Z \subseteq H^1(\Omega)$ is a finite dimensional subspace, then there exists $\varrho = \varrho(Z) > 0$ such that*

$$
\varphi\big|_{Z \setminus (Z \cap B_\varrho)} \leqslant 0.
$$

Proof. For $u \in Z \subseteq H^1(\Omega)$, we have the unique sum decomposition

$$
u = \overline{u} + u^0 + \widehat{u},
$$

with $\overline{u} \in H_-$, $u^0 \in E(0)$, $\widehat{u} \in H_+$. Exploiting the orthogonality of the component space in this decomposition, we have

$$
\begin{aligned}
\varphi(u) &= \frac{1}{2}\gamma(\overline{u}) + \frac{1}{2}\gamma(\widehat{u}) - \int_\Omega F(z,u)\,dz \\
&\leqslant \frac{1}{2}\gamma(\overline{u}) + \frac{1}{2}\gamma(\widehat{u}) - \eta\|u\|_2^2 + c_{21} \\
&\leqslant \frac{1}{2}\gamma(\widehat{u}) - \eta\|\widehat{u}\|_2^2 - \eta\|\overline{u}\|_2^2 - \eta\|u^0\|_2^2 + c_{21}
\end{aligned}
$$

$$\leqslant \quad -c_{22}\big(\|\widehat{u}\|^2 + \|\overline{u}\|^2 + \|u^0\|^2\big) + c_{21} \;=\; -c_{22}\|u\|^2 + c_{21}$$

for some $c_{21}, c_{22} > 0$ by choosing $\eta > 0$ big enough (see (3.53), use the Pythagorean theorem and the fact that $\overline{u} \in H_-$). So, we can find $\varrho > 0$ big enough such that

$$\varphi\big|_{Z\setminus(Z\cap B_\varrho)} \leqslant 0.$$

□

Now we are ready for the multiplicity theorem.

Theorem 4.4. *If hypotheses $H(\xi)$, $H(\beta)$ and H_2 hold, then there exists a sequence of nontrivial solutions $\{u_n\}_{n\geqslant 1} \subseteq C^1(\overline{\Omega})$ of (1.1) such that $\|u_n\| \longrightarrow +\infty$.*

Proof. Propositions 4.2 and 4.3 permit the use of Theorem 2.4. Since φ maps bounded sets to bounded sets, according to Theorem 2.4, we can find a sequence $\{u_n\}_{n\geqslant 1} \subseteq H^1(\Omega)$ such that

$$\{u_n\}_{n\geqslant 1} \subseteq K_\varphi \setminus \{0\}, \quad \|u_n\| \longrightarrow +\infty.$$

Hence the u_n's are nontrivial solutions of (1.1) and the regularity theory of Wang [22] implies that

$$\{u_n\}_{n\geqslant 1} \subseteq C^1(\overline{\Omega}).$$

□

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