Optimal factors in Vladimir Markov's inequality in L2 Norm

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Abstract
In this paper we discuss a problem of computation of constants in Vladimir Markov's type inequality in L^2 norm on the interval [–1; 1].

Key words:
V. Markov's inequality, L2 norms

1. Vladimir Markov's inequality.

The famous V. Markov's inequality is the following bound for a polynomial P in one variable with real coefficients of degree at most n

$$\sup_{x\in[-1,1]} |P^{(k)}(x)| \le \frac{n^2(n^2-1)\cdots(n^2-(k-1)^2)}{1\cdot 3\cdots(2k-1)} \sup_{x\in[-1,1]} |P(x)|$$
$$= T_n^{(k)}(1) \sup_{x\in[-1,1]} |P(x)|,$$

where T_n denotes the n-th Chebyshev polynomial of the first kind, which is given by the formula $\cos(nx) = T_n(\cos x)$. The case $k = 1$ was firstly considered by Dmitrij Mendeleev (yes, the famous Russian chemist!) and Andriey Markov, and for this reason the above inequality is known as *Markov's inequality*.

In the sequel, we shall write $\sup_{x\in[-1,1]} |P(x)| =: ||P||_\infty$ and for $1 \le p < \infty$

$$||P||_p = \left(\frac{1}{2}\int_{-1}^{1} |P(x)|^p dx\right)^{1/p}.$$

We can rewrite V. Markov's inequality in the forms

$$||P^{(k)}||_\infty \le ||T_n^{(k)}||_\infty ||P||_\infty$$

or

$$||P^{(k)}||_\infty \le V(n,k)n^{2k}||P||_\infty,$$

where $V(n,k) = \frac{1}{(2k-1)!!}\left(1-\frac{1}{n^2}\right)\cdots\left(1-\frac{(k-1)^2}{n^2}\right)$.

The above inequalities imply the following facts

- $V(n+1,k) \le V(n,k),\ n \ge k$.
- $\lim_{n\to\infty} V(n,k+1)/V(n,k) = \frac{1}{2k-1}$.

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- There exists a positive constant C such that for any polynomial P of degree at most n,

$$||P^{(k)}||_\infty \leq C^k \frac{1}{k!} n^{2k} ||P||_\infty .$$

- There exists a family of monic polynomials $(\widehat{P}_n)$ such that $\widehat{P}_n(-x) = (-1)^n \widehat{P}_n(x)$ and

$$||\widehat{P}_n||_\infty = \inf\{||Q||_\infty : \ Q \text{ is a monic polynomial of degree } n\}$$

$$||\widehat{P}_n^{(k)}||_\infty / ||\widehat{P}_n||_\infty = \sup\{||Q^{(k)}||_\infty / ||Q||_\infty : \ \deg Q \leq n\}.$$

- In particular,

$$||\frac{d^k}{dx^k}\widehat{P}_n||_\infty / ||\widehat{P}_n||_\infty = \sup\{||\frac{d^k}{dx^k}\widehat{P}_n^{(a,a)}||_\infty / ||\widehat{P}_n^{(a,a)}||_\infty : \ a > -1\}.$$

Here $(\widehat{P}_n^{(a,a)})$ is the family of monic ultraspherical polynomials belonging to the larger family of monic Jacobi polynomials $\widehat{P}_n^{(\alpha,\beta)}$.

2. Markov's inequality for the first and the second derivative in L_p norms.

For the first derivative of polynomials, and $p \geq 1$ there exists a positive constant C_p such that

$$||P'||_p \leq C_p \cdot (\deg P)^2 ||P||_p .$$

This inequality was firstly proved in [18], motivated by Zygmund's inequality in [36]. In the special case $p = 2$ it was proved by E. Schmidt ([26],[27]) that $C_2 = \sqrt{3}$. He also obtained a remarkable result: in the inequality $||P'||_2 \leq A(\deg P)||P||_2$ we have

$$\lim_{n\to\infty} A(n)/n^2 = \frac{1}{\pi}.$$

After Hille, Schegö, Tamarkin and Schmidt, Markov's inequality in L_p norms (and its generalizations with various weights) was investigated by a number of specialist, especially at the end of twentieth century, when the case $p = 2$ was thoroughly studied, cf. [2], [3],[5], [7],[9], [10], [11], [12], [13], [14], [15], [16], [17], [19], [20], [21], [22], [23], [28], [29], [31], [32], [33], [35]. Sixty years after [18] it was proved in [5] that $\lim_{p\to\infty} C_p = 1$. Thus the classical A. Markov's inequality is the limit case from [18]. It was conjectured by P. Goetgheluck in [16] that $C_p = (p+1)^{1/p}$, which agrees with known values for $p = 2, \infty$. Many authors are searching for the best estimates of the type $||P^{(k)}||_p \leq C_p(n,k)||P||_p$, $\deg P \leq n$ or try to describe the asymptotic behavior of $C_p(n,k)$. It was completely solved in the case $p = 2$:

- $\lim_{n\to\infty} C_2(n,1)/n^2 = \frac{1}{\pi}$, E. Schmidt ([26],[27]);
- $\lim_{n\to\infty} C_2(n,2)/n^4 = \frac{1}{4k_0^2} = 0.0711..$, where $k_0 = \inf\{k > 0 : \ 1 + \cos k \cosh k = 0\}$, L. Shampine ([28],[29]);
- $\lim_{n\to\infty} \frac{C_2(n,k)}{n^{2k}} = ||L_k^*||_{L^2(0,1)}$, where $L_k^* f(x) = \frac{1}{2^k(k-1)!}\int_0^x (x-y)^{k-1} f(y)dy$, A. Böttcher, P. Dörfler ([9]).

Remark 2.1. Let us observe that $\lim_{n\to\infty} V(n,k)/n^{2k} = 1/(2k-1)!!$ and

$$\lim_{k\to\infty} \left(2^k k!/(2k-1)!!\right)^{1/k} = 1.$$

This suggests a connection between limits $\lim_{n\to\infty} C_p(n,k)$ and the L_p norms of Volterra's operators

$$L_k^* f(x) = \frac{1}{2^k(k-1)!}\int_0^x (x-y)^{k-1} f(y)dy.$$

By applying Schwarz inequality, it is easy to get the following upper bound

$$||L_k^*||_{L_2(0,1)} \le \frac{1}{2^k(k-1)!}\frac{1}{\sqrt{(2k-1)2k}}.$$

It gives for $k = 2$ the upper bound $\frac{1}{8\sqrt{3}} = 0.0721....$ while the exact value is 0.0711....

Moreover, the lower bound may be found in [9]

$$||L_k^*||_{L_2(0,1)} \ge \frac{1}{2^k(k-1)!}\frac{1}{\sqrt{(2k-1)(2k+1)}}.$$

In particular, for $k = 3$, we get

$$0.010564... \le ||L_3^*||_{L_2(0,1)} \le 0.011410...$$

3. Markov's inequality for third derivative in L_2 norm.

Refining a method from [5], G. Sroka [32] obtained the following non-trivial result.

Theorem 3.1. *If $p \ge 1$, then for an arbitrary polynomial P of degree $k \le \deg P \le n$ we have inequality*

$$||P^{(k)}||_p \le (C(p+1)k^2)^{1/p} ||T_n^{(k)}||_\infty ||P||_p,$$

with $C = 6\sqrt[3]{4}e^2$ for $k \ge 3$.

Later M. Baran and P. Ozorka (see P.Ozorka's PhD thesis) proved, applying quite different technique, the following theorem.

Theorem 3.2. *If $1 \leq p \leq 2$, then for an arbitrary polynomial P of degree $k \leq \deg P \leq n$ we have inequality*

$$||P^{(k)}||_p \leq B_p \max_{k\leq l\leq n} ||T_l^{(k)}||_p n^{\frac{2}{p}} ||P||_p = B_p ||T_n^{(k)}||_p n^{\frac{2}{p}} ||P||_p,$$

with $B_p = (3e/\pi)^{1/p}(p+1)^{1/p}$.

Corollary 3.3. *If $1 \leq p \leq 2$ is fixed then there exists a constant C_p independent of n and k such that for all $k \geq 3$ the following Vladimir Markov type inequality holds*

$$||P^{(k)}||_p \leq C_p^k \frac{1}{k!} n^{2k} ||P||_p.$$

Remark 3.4. Applying Nikolski inequality (cf. [32] Lemma 3 where a little strong result is given) $||P||_\infty \leq (2p+2)^{1/p}(\deg P)^{2/p}||P||_p$, we derive from Theorem 3.1 the following

$$||P^{(k)}|| \leq (C(p+1)k^2)^{1/p}(2p+2)^{1/p}n^{2/p}||T_n^{(k)}||_p||P||_p, \ \deg P \leq n,$$

which is considerably worse than Theorem 3.2.

Now we shall discuss the case $p = 2$ to compare Theorems 3.1 and 3.2 with earlier known results.

Denote

$$B(n,k) = (3e/\pi)^{1/2}\sqrt{3}n \cdot ||T_n^{(k)}||_2,$$

$$A(n,k) = (6\sqrt[3]{4}e^2)^{1/2}\sqrt{3}k \cdot ||T_n^{(k)}||_\infty$$

and

$$R(n,k) = \frac{A(n,k)}{B(n,k)}.$$

We have

$$R(n,k) = (2e\sqrt[3]{4}\pi)^{1/2}(k/n)||T_n^{(k)}||_\infty / ||T_n^{(k)}||_2 \approx 5.20692 \cdot \rho_n^{(k)},$$

where

$$\rho_n^{(k)} = (k/n) \cdot ||T_n^{(k)}||_\infty / ||T_n^{(k)}||_2.$$

n	$\|\|T_n^{(3)}\|\|_2$	$n\cdot\|\|T_n^{(3)}\|\|_2$	$\|\|T_n^{(3)}\|\|_\infty$	$3\cdot\|\|T_n^{(3)}\|\|_\infty$	$\rho_n^{(3)}$
3	24	72	24	72	1
4	110.851	443.405	192	576	1.299
5	349.17	1745.85	840	2520	1.443
6	882.842	5297.05	2688	8064	1.522
7	1926.46	13585.2	7056	21168	1.570
8	3779.46	30235.7	16128	48344	1.599
9	6840.29	61562.6	33264	99792	1.621
10	11620.5	116205	63360	190080	1.636
11	18759.1	206350	113256	339768	1.647
12	29036.1	348434	192192	576576	1.655
13	43387.6	564039	312312	936936	1.661
14	62919.1	880867	489216	1467648	1.666
15	88919.9	1333800	742560	2227680	1.670
16	121878	1966040	1096704	3290112	1.673
17	166491	2830360	1581408	4744224	1.676
18	221687	3990370	2232576	6697728	1.678
19	290632	5522000	3093048	9279144	1.680
20	375745	7514910	4213440	12640320	1.682
30	2859070	85772200	48330240	144990720	1.690
40	12056700	482268000	272213760	816641280	1.693
50	36806500	1840325000	103958400	3118752000	1.695
100	11783330000	11783000000	6663333600	19990008000	1.697

$$\rho_n^{(3)} = (3/n)\cdot ||T_n^{(3)}||_\infty / ||T_n^{(3)}||_2$$

n	$\|\|T_n^{(4)}\|\|_2$	$n\cdot\|\|T_n^{(4)}\|\|_2$	$\|\|T_n^{(4)}\|\|_\infty$	$4\cdot\|\|T_n^{(4)}\|\|_\infty$	$\rho_n^{(4)}$
10	178306	1783060	823680	3294270	1.848
20	$241168\cdot 10^2$	$483362\cdot 10^3$	235350720	941402880	1.948
30	$417336\cdot 10^3$	$1252008\cdot 10^3$	6151749120	2460697965	1.965
40	$313811\cdot 10^4$	$1.25524\cdot 10^{11}$	61870298880	$2.47481\cdot 10^{11}$	1.972
50	$1.49894\cdot 10^{10}$	$7.47795\cdot 10^{11}$	$3699943392\cdot 10^3$	$1.479773560\cdot 10^{12}$	1.979

$$\rho_n^{(4)} = (4/n)\cdot ||T_n^{(4)}||_\infty / ||T_n^{(4)}||_2.$$

Analyzing the above numerical results we see that the bounds in Theorem 3.2 are much better than the bounds in Theorem 3.1. We can also suppose that the factor $k^{2/p}$ in Theorem 3.1 can be replaced by $k^{1/p}$. Moreover, we can conjecture that

- $(k/n)\cdot ||T_n^{(k)}||_\infty / ||T_n^{(k)}||_2 \le \sqrt{k}$,
- $(k/n)\cdot ||T_n^{(k)}||_\infty / ||T_n^{(k)}||_2 \nearrow \sqrt{k}$ as $n \to \infty$.

Further calculations.

$$C(n,1) = \sup_{\deg P = n} \frac{1}{n^2} ||P'||_2/||P||_2$$

$$A(n,1) = V(n,1) = \frac{1}{n^2} \sup \left\{ \left\| \frac{dP_n^{(a,a)}}{dx} \right\|_2 /||P_n^{(a,a)}||_2 : \ a \geq 0 \right\}$$
$$= \frac{1}{n^2} \left\| \frac{dP_n^{(\alpha_n,\alpha_n)}}{dx} \right\|_2 /||P_n^{(\alpha_n,\alpha_n)}||_2.$$

n	α_n	$A(n,1)$	$C(n,1)$
1	0	1.732050	1.732050
2	0	0.968246	0.968246
3	0.133222	0.724622	0.724622
4	0.242328	0.60736	0.609363
5	0.325474	0.53958	0.543656
6	0.388334	0.49587	0.501657
7	0.436555	0.465519	0.472648
8	0.474328	0.443287	0.451468
9	0.504555	0.426332	0.435350
10	0.529225	0.41299	0.422685
11	0.549714	0.402224	0.412476
12	0.566993	0.393358	0.404076
13	0.581761	0.385931	0.397045
14	0.594533	0.37962	0.391075
15	0.605691	0.374191	0.385944
16	0.615528	0.369473	0.381486
17	0.62427	0.365333	0.377578
18	0.632094	0.361671	0.374124
19	0.639143	0.358417	0.371050
20	0.645528	0.355441	0.368296
21	0.651342	0.353229	0.365814
22	0.656662	0.350198	0.363567
23	0.66155	0.348284	0.361523
24	0.666058	0.346296	0.359655
25	0.670232	0.344458	0.357942
26	0.674108	0.342822	0.356365
27	0.677718	0.341360	0.354908
28	0.68109	0.335661	0.353559
29	0.684249	0.339102	0.352305
30	0.687214	0.309052	0.351138

It seems that the sequence $A(n,1)$ is decreasing. If it is true, the sequence $A(n,1)$ is convergent, but it is not clear that the limit is $1/\pi = 0.31830...$.

$$V(n,k) = \frac{1}{n^{2k}} \sup \left\{ \left\| \frac{d^k P_n^{(a,a)}}{dx^k} \right\|_2 / ||P_n^{(a,a)}||_2 : \; a \geq 0 \right\}$$
$$= \frac{1}{n^{2k}} \left\| \frac{d^k P_n^{(\alpha_n^{(k)},\alpha_n^{(k)})}}{dx^k} \right\|_2 / \left\| P_n^{(\alpha_n^{(k)},\alpha_n^{(k)})} \right\|_2 ,$$

$$||U_{n-1}^{(k)}||_2^* = ||U_{n-1}^{(k)}||_2 / n^{2k}.$$

n	$\alpha_n^{(2)}$	$V(n,2)$	$\alpha_n^{(3)}$	$V(n,3)$	$\|\|U'_{n-1}\|\|_2^*$	$\|\|U''_{n-1}\|\|_2^*$
1	0		0			
2	0		0			
3	0		0		0.5132	0.0987654
4	0.0626545	0.21803	0	0.044401	0.5123	0.108253
5	0.126775	0.181934	0.0354246	0.0373206	0.510283	0.111735
6	0.182076	0.159283	0.075835	0.0325326	0.508472	0.113534
7	0.227988	0.143887	0.113921	0.0290943	0.507044	0.114622
8	0.266016	0.132788	0.148038	0.0266326	0.505933	0.11534
9	0.297795	0.124424	0.17816	0.0246069	0.506062	0.115841
10	0.32467	0.117902	0.204704	0.0230643	0.504371	0.11205
11	0.347674	0.112677	0.228163	0.0218158	0.503813	0.116479
12	0.36759	0.1084022	0.248998	0.0207856	0.503358	0.11669
13	0.385008	0.104829	0.267603	0.0199217	0.502982	0.116856
14	0.400382	0.101808	0.28431	0.0191873	0.502668	0.116988
15	0.414062	0.0992172	0.299393	0.0185554	0.502402	0.117096
16	0.426323	0.0969702	0.31308	0.0180062	0.502175	0.117185
17	0.437383	0.0950031	0.325558	0.0175245	0.501979	0.117259
18	0.447419	0.0932664	0.336984	0.0170768	0.50181	0.117322
19	0.456573	0.0917213	0.34749	0.0167193	0.501663	0.117375
20	0.464961	0.0903384	0.357186	0.016380	0.501533	0.11742
21	0.47268	0.0890925	0.366166	0.0160731	0.501418	0.11746
22	0.479812	0.0879644	0.374509	0.0157956	0.501316	0.117494
23	0.486425	0.0869379	0.382284	0.0155430	0.501226	0.117524
24	0.492576	0.085999	0.38955	0.0153065	0.501144	0.11755
25	0.498316	0.0851355	0.396358	0.0151038	0.501071	0.117583
26	0.503685	0.0843456	0.402751	0.0148929	0.501005	0.117594
27	0.508722	0.0835003	0.408769	0.0147338	0.500945	0.117612
28	0.513457	0.0825563	0.414445	0.0144975	0.50089	0.117629
29	0.51792	0.0826230	0.41981	0.0144838	0.50084	0.117644
30	0.522134	0.0806537	0.424889	0.0142467	0.500795	0.117657

From the above numerical data one can conjecture that

$$\lim_{n\to\infty} V(n,k) = \lim_{n\to\infty} C_2(n,k).$$

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