Markov inequality on the graph of holomorphic function

Tomasz Beberok*

Department of Applied Mathematics University of Agriculture in Krakow, ul. Balicka 253c, 30-198 Kraków

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Abstract
The purpose of this paper is to show that the Markov inequality does not hold on the graph of holomorphic function.

Key words: Markov inequality; graph of holomorphic function; pluripolar sets

1 Introduction

A few years after chemist Mendeleev published his periodic table he made a study of the specific gravity of a solution as a function of the percentage of the dissolved substance. Mendeleev's study led to the following mathematical problem: estimate how large can be $|P'(x)|$ on $-1 \leq x \leq 1$ for a quadratic polynomial $P(x) = ax^2+bx+c$ with $|P(x)| \leq 1$ for $x \in [-1, 1]$ (for details, how the Mendeleev's problem in Chemistry amounts to this mathematical problem in polynomials, see [1]). Note that Mendeleev himself was able to solve this mathematical. Mendeleev told his result to a Russian mathematician A.A. Markov, who naturally investigated the corresponding problem in a more general setting, that is, for polynomials of arbitrary degree n. He [2] proved the following result which is now known as Markov inequality.

Theorem 1.1 *Let $P(x) = \sum_{k=0}^{n} a_k x^k$ be a real polynomial of degree n and $\|P(x)\|_{[-1,1]} \leq 1$ ($\|\cdot\|_K$ is the maximum norm on K). Then*

$$|P'(x)| \leq n^2 \quad \text{for} \quad -1 \leq x \leq 1. \tag{1}$$

This result is best possible since for the Chebyshev polynomials $T_k(x) = \cos k \arccos x$ ($x \in [-1, 1]$), $k = 1, 2, \ldots,$ of degree k one has $\|T_k\|_{[-1,1]} = 1$ and $|T_n'(\pm 1)| = n^2$.

Markov's inequality became soon a fascinating object of investigations. The reason lay with its numerous applications in different domains of mathematics and physics. Various analogues of the above Theorem are known in which the underlying intervals, the maximum norms, and the family of functions are replaced by more general sets, norms, and families of functions, respectively. These inequalities are called Markov-type inequalities. Markov-type inequalities are known on various regions of the complex plane and the N-dimensional Euclidean space, for various norms such as weighted L^p norms, and for many classes of functions such as polynomials with various constraints, exponential sums of n terms, just to mention a few. Several papers have been published in this area (see [3, 4, 5, 6, 7, 9, 10, 13, 16, 17, 18, 19, 20, 21, 22, 26, 27, 28, 33, 34, 36]), and it is not possible to include all of them here.

*Corresponding author: tbeberok@ar.krakow.pl

In the sequel, a compact set $E \subset \mathbb{K}$ ($\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$) is said to preserve (or admit) Markov's inequality, or simply to be Markov, if there exist constants $M > 0$ and $r > 0$ such that for each polynomial P in $\mathbb{R}^N$ we have

$$\|D^\alpha P\|_E \le M(\deg P)^r \|P\|_E, \quad \text{for every} \quad \alpha = (\alpha_1, \ldots, \alpha_N) \in \mathbb{N}_0^N, \tag{2}$$

where $D^\alpha := \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \ldots \partial x_N^{\alpha_N}}$ and $|\alpha| := \alpha_1 + \ldots + \alpha_N$. Sets with this property play an important role in the constructive theory of functions, especially in problems of polynomial approximation and extension of C^∞ functions (see e.g. [12, 14, 24, 25, 29, 35]).

2 The Bernstein-Walsh-Siciak theorem

A function $f\colon G \to \mathbb{R} \cup \{-\infty\}$ with domain $G \subset \mathbb{C}^N$ is called plurisubharmonic if it is upper semi-continuous, and for every complex line $\{a + bz\colon z \in \mathbb{C}\} \subset \mathbb{C}^N$ with $a, b \in \mathbb{C}^N$ the function $z \mapsto f(a + bz)$ is a subharmonic function on the set $\{z \in \mathbb{C}\colon a + bz \in G\}$ (for more see [23]).
For a compact set $K \subset \mathbb{C}^N$, we define

$$V_K(z) := \max\left\{0, \sup_P \left\{\frac{1}{\deg P} \log |P(z)|\right\}\right\}$$

where the supremum is taken over all non-constant polynomials P with $\|P\| \le 1$. This is a generalization of the one-variable Green function. The function V_K is lower semicontinuous, but it need not be upper semicontinuous. The upper semicontinuous regularization

$$V_K^*(z) = \limsup_{\zeta \to z} V_K(\zeta)$$

of V_K is either identically $+\infty$ or else V_K^* is plurisubharmonic. The first case occurs if the set K is too "small"; precisely if K is pluripolar: this means that there exists a plurisubharmonic function u defined in a neighborhood of K with $K \subset \{z\colon u(z) = -\infty\}$. We say that K is L-regular if $V_K = V_K^*$, that is, if V_K is continuous.
If the compact set $K \subset \mathbb{C}^n$ is L-regular, then for each $R > 1$ we define the set

$$D_R := \{z\colon V_K(z) < \log R\}. \tag{3}$$

Now we are ready to formulate a famous Bernstein-Walsh-Siciak theorem (see [31]).

Theorem 2.1 *Let K be an L-regular compact set in $\mathbb{C}^N$. Let $R > 1$, and let D_R be defined by (3). Let f be continuous on K. Then*

$$\limsup_{n \to \infty} d_n(f, K)^{1/n} \le 1/R$$

if and only if f is the restriction to K of a function holomorphic in D_R, where

$$d_n(f, K) := \inf\{\|f - P\|_K\colon \deg P \le n\}.$$

3 Main result

It is known that the Hölder continuity of V_K implies the Markov property of K (see [32]) and every set which admit Markov inequality seems to be L-regular but that has been proved so far only for compact subsets of $\mathbb{R}$ (see [11]). In the general (complex) case the question about the L-regularity of sets with the global Markov property remains an open problem posed by Pleśniak in [29]. However, every compact set $K \subset \mathbb{C}$ with the Markov property is not polar [8], which is a necessary condition for the continuity of the Green function (see e.g. [30] Theorems 4.4.2,3). Our main result is related to the following open problem: Does Markov inequality (2) imply that E is nonpluripolar?

Theorem 3.1 *Let K be a compact subset of $\mathbb{R}$. Let $f\colon K \to \mathbb{R}$ be the restriction to K of the holomorphic function defined on $D_R := \{z\colon V_K(z) < \log R\}$ for some $R > 1$. Then a graph $\Gamma_f := \{(z, f(z))\colon z \in K\}$ of f does not admit Markov inequality.*

Proof. Suppose, seeking a contradiction, that Γ_f admits the Markov inequality. There are now two cases:
Case1: The set K satisfies a Markov inequality. Hence K is L-regular. By the Bernstein-Walsh-Siciak theorem there exist a sequence $\{p_n\}$ of polynomials such that $\lim_{n\to\infty} \|f - p_n\|_K^{1/n} \le 1/R$. Now consider the sequence of polynomials $P_n(x, y) = y - p_n(x)$. It is clear that

$$\left\| \frac{\partial P_n}{\partial y} \right\|_K = 1.$$

However,

$$\|P_n\|_K \le (1/R)^n \quad \text{if } n \text{ is large enough.}$$

Therefore for every constants $M > 0$ and $r > 0$

$$M(\deg P_n)^r \|P_n\|_K \to 0 \quad \text{as} \quad n \to \infty.$$

This gives a contradiction, and the result is established.
Case2: The set K does not have a Markov property. In this case to get the contradiction it is enough to take one variable polynomials.
Example. Let us consider the following set

$$K := \{(x, e^x)\colon x \in [0, 1]\}.$$

For this set it is enough to take

$$p_k(x, y) = y - \sum_{n=0}^{k} \frac{x^n}{n!}.$$

Then

$$\|p_k\|_E = \left\| e^x - \sum_{n=0}^{k} \frac{x^n}{n!} \right\|_{[0,1]} = \left\| \sum_{n=k+1}^{\infty} \frac{x^n}{n!} \right\|_{[0,1]} = \sum_{n=k+1}^{\infty} \frac{1}{n!} = e - \frac{e\Gamma(k+1, 1)}{\Gamma(k+1)},$$

where

$$\Gamma(s,x) = \int_x^\infty t^{s-1}e^{-t}\,dt.$$

Let

$$\gamma(s,x) = \int_0^x t^{s-1}e^{-t}\,dt.$$

Hence

$$\Gamma(s,x) = \Gamma(s) - \gamma(s,x).$$

Therefore

$$\|p_k\|_E = e\frac{\gamma(k+1,1)}{\Gamma(k+1)}.$$

And now, using (5.4) from [15] (see page 19), we have

$$\|p_k\|_E = e\frac{\gamma(k+1,1)}{\Gamma(k+1)} < e(1-1/e)^{k+1}.$$

This gives a contradiction with the Markov inequality for the set K.

Now a similar proof to that of the last theorem gives the following generalization:

Theorem 3.2 *Let K be a L-regular subset of $\mathbb{C}^N$. Let $f\colon K \to \mathbb{C}$ be the restriction to K of the holomorphic function defined on $D_R := \{z\colon V_K(z) < \log R\}$ for some $R > 1$. Then a graph $\Gamma_f := \{(z, f(z))\colon z \in K\}$ of f does not admit Markov inequality.*

Note that each graph Γ_f is a pluripolar set. Therefore above theorem is a partially solution to the difficult problem whether Markov property implies nonpluripolarity (if $N > 1$).

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Tomasz Beberok
Department of Applied Mathematics
University of Agriculture in Krakow
ul. Balicka 253c, 30-198 Kraków
email: tbeberok@ar.krakow.pl